m odd $\qquad n = m+1$ even.

A family of projective hypersurfaces:

$$t \in \mathbb{P}^1 \qquad Y_t \subset \mathbb{P}^n$$

$$Y_t : X_0^{n+1} + \cdots + X_n^{n+1} = (n+1)\, t\, X_0 \cdots X_n$$

<u>Smooth</u> when $t \in T = \mathbb{P}^1 - (\infty \cup \mu_{n+1})$.

On Y_t we have an action of

$$H = \{ (\eta_0, \cdots, \eta_n) \in \mu_{n+1}^{n+1} \;/\; \prod \eta_i = 1 \}$$

Define V_t, resp. $V[N]_t$, resp. $V_{\ell,t}$, resp $V_{DR,t}$ to be the set of H-invariants in the cohomology groups:

$$H^{n-1}(Y_t, -)^H \qquad - = \mathbb{Z},\ \mathbb{Z}/N\mathbb{Z}, \ \mathbb{Q}_\ell,\ DR$$

Facts

- for $t \in T$ those are rank n modules (or vector spaces)
- We have a Galois action on $V[N]_t$ and $V_{\ell,t}$ for $t \in \mathbb{Q}$ a algebraic.
- The Hodge numbers on V_{DR} are $0, 1, \dots, n-1$.

Moreover because $n-1$ is odd we have a perfect alternating form

$$V_t \times V_t \longrightarrow \mathbb{Z}$$
$$V[N]_t \times V[N]_t \longrightarrow (\mathbb{Z}/N\mathbb{Z})(1-n)$$
$$V_{\ell,t} \times V_{\ell,t} \longrightarrow \mathbb{Q}_\ell(1-n)$$

Suppose we fix some $t \in \mathbf{T}$. The fundamental group $\pi_1(T,t)$ acts (monodromy operation) on V_t and this action is compatible with the simplectic form.

So we get an homomorphism:

$$\pi_1(T,t) \longrightarrow Sp(V_t) \quad (\simeq Sp(n,\mathbb{Z}))$$

A key theorem in [HSBT] is the following:

<u>Theorem</u> <u>The image of the above isomorphism is Zariski-dense in</u> $Sp(V_t \otimes \mathbb{C})$ $(\simeq Sp_n(n,\mathbb{C}))$

The proof given in [HSBT] consists in a direct computation of the monodromy around the points of $(\infty \cup \mu_{n+1})$ where the family is not smooth. It also uses some results on hypergeometric groups (Beukers and Heckman).

Using some deep congruence properties of Zariski-dense subgroups due to several people (Nori, Matthews-Vaserstein-Weisfeiler) it is possible to deduce from the above theorem the following one:

<u>Theorem</u> <u>There is a constant</u> $C(n)$ <u>such that if all prime factors of N are</u> $> C$ <u>then the homomorphism</u>

$$\pi_1(T,t) \longrightarrow Sp(V[N]_t)$$

<u>is surjective</u>.

That means philosophically that the family $V[N]_t$, when t varies in T, is "as far as being constant as possible". So it provides "many" mod N representations.

This theorem is used in connection with a theorem of Moret-Bailly which asserts that, given a geometrically irreducible scheme over a number field F, it has a point over some Galois extension F'/F, satisfying some local conditions, provided it is true locally.

<u>Th</u> [MB] Suppose given a number field F, and Z/F a smooth and <u>geometrically connected</u> variety. Let $S = S_1 \sqcup S_2$ a finite set of places in F, such that S_2 contains only finite places. For each $v \in S_1$ an open non empty set $\Omega_v \subset Z(F_v)$ is given; also for $v \in S_2$ we have an open non empty $\Omega_v \subset Z(F_v^{nr})$ stable under $\mathrm{Gal}(F_v^{nr}/F_v)$. The last datum is a finite extension L/F.

Then there exists a finite Galois F'/F linearly disjoint from L and a point $P \in Z(F')$ such that:

- each $v \in S_1$ is completely split in F', and for w a place of F' above v ($F'_w \simeq F_v$) we have $P \in \Omega_v \subset Z(F'_w)$.

- each $v \in S_2$ is unramified in F' and for w above v ($F'_w \subset F_v^{nr}$, well defined up to $\mathrm{Gal}(F_v^{nr}/F_v)$) we have $P \in \Omega_v \cap Z(F'_w)$.

Here is a rough idea of how those theorems are applied in our context. Suppose $\ell \neq \ell'$ $> C$ are two primes and that $\bar{\sigma}_\ell$ is a given modulo ℓ symplectic Galois representation <u>we want to prove</u> to be automorphic.

On the other hand assume that $\bar{\sigma}'_{\ell'}$ is a modulo ℓ' symplectic Galois representation <u>we know</u> to be automorphic.

Typically $\bar{\sigma}'_{\ell'}$ can be some induced representation from a well-chosen character of a degree n extension.

On $T_{\mathbb{Q}}$ (T viewed as a line over $\mathbb{Q}$) we have two symplectic local systems: One is $V[\ell]_t \times V[\ell']_t = V[\ell\ell']_t$ and the other comes from $\operatorname{Spec} \mathbb{Q}$ where $\bar{\sigma}_\ell \times \bar{\sigma}'_{\ell'}$ defines a local system.

Define

$$Z = \underline{\mathrm{Isom}}_T^{\,\mathrm{Sym}}\left(\underline{\bar\sigma_\ell \times \bar\sigma_{\ell'}} , \underline{V[\ell\ell']} \right)$$

This is defined over $\mathbb{Q}$ and an étale covering of T. In down to earth term, to give a point of Z over some algebraic number field F' amount to the datum

of $\quad t \in T(F') = F' \smallsetminus \mu_{n+1}$

and of two symplectic isomorphisms compatible with $\mathrm{Gal}(\bar{\mathbb{Q}}/F')$:

$$\bar\sigma_\ell \xrightarrow{\sim} V[\ell]_t \qquad \bar\sigma_{\ell'} \xrightarrow{\sim} V[\ell']_t .$$

Over $\mathbb{C}$ the sheaf $\bar\sigma_\ell \times \bar\sigma_{\ell'}$ becomes trivial so $Z_{\mathbb{C}}$ can be identified with

$$\underline{\mathrm{Isom}}_{T_{\mathbb{C}}}^{\,\mathrm{Sym}}\left(\mathbb{Z}/\ell\ell'\mathbb{Z} , \underline{V[\ell\ell']} \right)$$

and this is an étale Galois covering of $\mathbb{C}$ with Galois group

$Sp(\mathbb{Z}/\ell\ell'\mathbb{Z})$. Because of the above theorem an the surjectivity of the monodromy we see that $Z_{\mathbb{C}}$ is connected.

Then the Moret-Bailly theorem can be applied, provided some local conditions are checked (.....)

$\hookrightarrow$ This gives F' $t \in F'$

and isomaphisms :

$$V[\ell]_t \simeq \bar{\sigma}_\ell \qquad \mathrm{Gal}(\bar{F}/F')\text{-}$$
$$V[\ell']_t \simeq \bar{\sigma}'_{\ell'} \qquad \text{inv}$$

Then (modulo a lot of hypothesis that are not so easy to check) we have:

$$\begin{array}{ccc} \bar{\sigma}'_{\ell'} \text{ aut} & \xRightarrow[\text{ALT}]{} & V[\ell']_t \text{ aut} \\ & & \Big\Downarrow \text{(compatible system).} \\ & & V[\ell]_t \text{ aut} \xRightarrow[\text{Red}]{} \bar{\sigma}_\ell \text{ aut.} \end{array}$$

This is the rough idea but cannot in general be applied directly to the representation $\bar\rho_m$ we are interested in.

We can fulfill the local condition at ℓ under the hypothesis that

$$\bar\sigma_\ell \,|\, I_{\mathbb{Q}_\ell} \simeq 1 \oplus \bar\varepsilon_\ell^{-1} \oplus \cdots \oplus \bar\varepsilon_\ell^{1-n}$$

cyclotomic

But in general we only have a filtration with subquotients $\bar\varepsilon_\ell^{-i}$: this is already the case for ρ_1.

One preliminary operation we have to perform is a "change of prime" at the level of the original elliptic curve E and then apply the idea above.

For this purpose one applies MB theorem to a certain space of elliptic curves to find another E' and ℓ'

such that:

- $H^1(E', \mathbb{F}_{\ell'}) = 1 \oplus \bar{\omega}_{\ell'}$ above condition.
- $H^1(E', \mathbb{F}_\ell) \simeq H^1(E, \mathbb{F}_\ell)$ as Gal. modules

(after maybe a field extension).

Because the above condition is fulfilled by E', ℓ' we have that

$$\mathrm{Sym}^m H^1(E', \mathbb{F}_{\ell'}) \text{ is automorphic.}$$
$$\Downarrow \text{ ALT}$$
$$\mathrm{Sym}^m H^1(E', \mathbb{Q}_{\ell'}) \text{ aut.}$$
$$\Downarrow \text{ compatible system}$$
$$\mathrm{Sym}^m H^1(E', \mathbb{Q}_\ell) \text{ aut.}$$
$$\Downarrow$$
$$\mathrm{Sym}^m H^1(E', \mathbb{F}_\ell) \text{ aut.}$$
$$\parallel$$
$$\mathrm{Sym}^m H^1(E, \mathbb{F}_\ell)$$

All this constitutes a sophisticated generalisation of Wiles' original change of prime method (3-5 or ℓ-ℓ' trick)

The aim of this last lecture is to give some hints about the proof of the automorphy lifting theorems.

▶ Original Taylor–Wiles method (improved by many people).

Start from $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{F}}_\ell)$ be a "modular" Galois representation, attached to a mod ℓ modular form $\bar{f}$.

There exists (Mazur) an universal ring $\underline{R}$ which is universal for deformations of $\bar{\rho}$ over artinian local rings A

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(A)$$
(modulo $\sim$).

Rather than this huge ring one considers more reasonable quotients R_S corresponding to representations satisfying some local conditions (in particular at ℓ) and with prescribed ramification S. extra ramification

On the other hand we consider $\mathbb{T}_S$ a Hecke algebra (associated to forms of a certain type and prescribed conductor) localized at a maximal ideal m corresponding to $\bar{\rho}$ (Ker. of $\begin{array}{l} \mathbb{T} \to \mathbb{F}_\ell \\ T_p \to \bar{a}_p \end{array}$)

Finally the classical Eichler-Shimura-Deligne construction (using modular curves) associates to each modular eigenform f a corresponding 2-dimensional ℓ-adic representation. This works also over rings and gives a representation

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{T}_S).$$

By universality of R_S we finally get a map

$$R_S \longrightarrow \mathbb{T}_S$$

which is easily seen to be surjective.
The problem is to prove that it is in fact bijective, provided of course that a series of hypothesis is satisfied.

The proof goes in two steps:

Step 1 Proof of the theorem in the "minimal" case $S = \phi$ which means that ρ is not more ramified than $\bar{\rho}$.

$$R_\phi \xrightarrow{\sim} \mathbb{T}_\phi$$

The proof goes as follows: for each N one adds a certain number of primes $p \in Q_N$ ($\equiv 1 \bmod \ell^N$) to kill relations at the limit

$$R_{Q_N} \longrightarrow \mathbb{T}_{Q_N}$$
$$R_\infty \xrightarrow{\sim} \mathbb{T}_\infty$$

essentially because we get smooth rings

of the same dimension.

<u>Step 2</u> minimal case $\to$ general case

using a lemma of Ihara which allows to control how things vary as the ramification increases.

In the first article [CHT] they followed the same ideas. Step 1 is ok but Step 2 needs a conjectural generalization of Ihara's lemma which was never proved.

Let me first explain Step 0, the construction of Galois representations and some related things.

Autodual ($\pi^\vee \simeq \pi \otimes \chi$) automorphic representations of GL_n *

(n dim ℓ-adic representations) *

"Base change"

Automorphic representations of forms of unitary groups **

of ∞ type $(1, n-1)$

Shimura varieties ($\simeq$ Picard varieties)

of ∞ type $(0, n)$ $\longrightarrow$ for the deformation theory

* Sometimes of a CM extension

** $B/E/\mathbb{Q}$ division algebra with an involution of the 2nd kind $\{ b b^* = 1 \}$ with a finite place such that B_q is compact.

$\mathbb{T}_S$ localized Hecke algebra as above (for a unitary group of type $(0,n)$)

R_S universal ring which classify autodual deformations of $\bar{\rho}$ satisfying certain conditions (in particular crystalline at ℓ) and with prescribed ramification.

As above we get a surjective map:

$$R_S \twoheadrightarrow \mathbb{T}_S$$

Step 1 $\quad R_\phi \xrightarrow{\sim} \mathbb{T}_\phi \quad$ OK

Step 2 $\quad$??

▶ Taylor's first new idea.

Instead of starting from the minimal case (best situation) he starts from the worst one, where the difference between the ramification of $\bar{\rho}$ and that of ρ is as big as possible ie

$\bar{\rho}$ unramified

ρ with unipotent ramification* at some places v $(q_v \equiv 1 \ (\ell))$.

* for $\sigma \in I_v$: $\rho(\sigma) = \exp(t_\ell(\sigma) N)$.

By base change (restriction) we can always reduce to this case.

<u>Local lifting problem</u> (ideas of Kisin)

Look at liftings of $\bar{\rho} \mid \mathrm{Gal}(\bar{F}_v / F_v)$.
Such a lifting is necessarily <u>tame</u> and so given by two matrices

ϕ image of Frobenius

Σ image of a generator of tame inertia

such that Σ is unipotent and such that

$$\phi \Sigma \phi^{-1} = \Sigma^{q_v}$$

There is a slight subtle difference between (global) <u>deformations</u> and (local) <u>liftings</u>: the first ones are given modulo conjugacy.

We have a very explicit ring $\mathcal{L}_v$ representing local liftings (as above), and a ring R representing global deformations.

Because of the subtlety mentionned above <u>there do not exist</u> a functor:

global deformations $\longrightarrow$ local liftings

To get such a thing we have to rigidify at v the global deformation ρ.

$R^{\square}$ classifying framed deformations.

Then we do have a map:

$$\bigotimes \mathcal{L}_v \longrightarrow R^{\square}$$

Corresponding objects $\mathbb{T}^{\square}$ $\mathcal{A}^{\square}$ (space of aut. forms) add Taylor-Wiles primes, go to the limit. We finally get the following picture:

$$\begin{array}{ccccc} & \text{Kisin} & & & \text{Diamond-Fujiwara} \\ \otimes \mathcal{L}_v[[z_1, \dots z_n]] \simeq & R_\infty^\square & (\longrightarrow T_\infty^\square) & \circlearrowright & A_\infty^\square \end{array}$$

One has to show that
$$\mathrm{Supp}\,(A_\infty^\square) = \mathrm{Spec}\, R_\infty^\square$$

It is possible to compute the dimension of this support but this is not enough because $\mathcal{L}_v$ (and $R_\infty^\square$) may have several irreducible components.

▶ <u>Taylor's second idea.</u>

Compare the above deformation rings with other ones.

instead of Σ unipotent $\longrightarrow$ Σ with ch. polynomial
$(X-1)^n = 0$ $\qquad$ $(X-\lambda_1)(X-\lambda_2)\cdots(X-\lambda_n)$

where $\lambda_1 \dots \lambda_n$ are distinct ℓ-th roots of 1.

With this new moduli pb we get rings
$\tilde{\mathcal{L}}_v \quad \tilde{R}^{\square}_\infty \quad \tilde{\mathbb{T}}_\infty \quad \tilde{A}^{\square}_\infty$.

Note that everything coincide mod. ℓ.

The following proposition is obtained by using a sophisticated version of Jordan reduction theory of matrices over rings. Even if it is not so easy to prove it is a statement in "classical" algebraic geometry.

<u>**Prop**</u> (a) Spec $\tilde{\mathcal{L}}_v$ is irreducible of dim. $1+n^2$.

(b) Each irreducible component of the special fiber Spec $(\mathcal{L}_v \otimes \mathbb{F}_\ell)$ is contained in a unique irreducible component of Spec $(\mathcal{L}_v)$. Conversely every component of Spec $(\mathcal{L}_v)$ contains an irreducible component of the special fiber.

Now it is time to conclude.

- Because of (a) things work for the rings $\tilde{*}$:
The support of $\tilde{\mathcal{A}}^{\square}_{\infty}$ is Spec $\tilde{R}_{\infty}$.

- Because the special fibers are the same,
Spec $(R_{\infty} \otimes \mathbb{F}_{\ell})$ is included in the support of $\mathcal{A}^{\square}_{\infty}$

- Using (b) one finally sees that the support is the whole of Spec (R_{∞}).
