Arithmetic Chow rings and arithmetic characteristic classes

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23-05-2005

The geometry of numbers

Arithmetic curves

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers.
The scheme $X = \operatorname{Spec} \mathcal{O}_K$ is an affine curve (we will call it an arithmetic curve) and its behaviour is similar to that of an affine curve defined over a field (a geometric curve).
We want to "compactify" X in the same way as an affine curve over a field can be compactified to yield a projective curve.
To this end we will start looking more closely at the geometric case.

The affine line

Let now $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. The function field of $\mathbb{A}^1$ is $\mathbb{C}(t)$.
We can compactify $\mathbb{A}^1$ adding one point at infinity ∞ and we write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$.
From an algebraic point of view, what interests us is whether a given rational function has a zero or a pole at a given point.
For any point $x \in \mathbb{A}^1$ there is a discrete valuation of $\mathbb{C}(t)$ denoted ord_x that gives us this information.
But there is another discrete valuation $\operatorname{ord}_\infty(f(t)) = \operatorname{ord}_0(f(1/t))$ that tells us exactly when the function f has a zero or a pole at the new point.

The points of $\mathbb{P}^1$ are in bijective correspondence with the set of valuations of $\mathbb{C}(t)$.

The compactified arithmetic curve

Following by analogy with the geometric case, we observe that, to every point $p \in X$, we can associate a discrete valuation of K, that tells us when an element $f \in K$ has a zero or a pole on the given point.

There is no other discrete valuations of K!.

To a given discrete valuation we can associate a norm

$$\|f\|_p = N(p)^{-\operatorname{ord}_p f}.$$

Besides the norms associated with discrete valuations, we find the Archimedean norms that are associated with non-equivalent complex immersions of K. Let S_∞ be the set of Archimedean norms.

The compactified arithmetic curve is $\overline{X} = X \cup S_\infty$.

The analogy between arithmetic and algebraic curves

Let Y be a projective geometric curve defined over $\mathbb{C}$.
The fact that Y is projective is reflected in the residue formula, that implies that, if $f \in K(Y)$ is a rational function then

$$\sum_{x \in Y} \operatorname{ord}_x f = 0.$$

The analogous statement for compactified arithmetic curves is the product formula, that says that, if $f \in K$, then

$$\prod_{p \in X} \|f\|_p \prod_{\nu \in S_\infty} \|f\|_\nu = 1.$$

Observation: With the right normalization we can use the set of complex immersions of K, Σ, instead of the set of Archimedean norms.

The geometric Riemann-Roch theorem

Let Y be a geometric projective curve. Let $\mathcal{L}$ be a line bundle over Y.
The Riemann-Roch theorem states that

$$\dim H^0(Y,\mathcal{L}) - \dim H^1(Y,\mathcal{L}) = \deg(\mathcal{L}) + 1 - g(Y).$$

One application of the Riemann-Roch theorem is a criterion for when a line bundle has global sections.

Theorem (Asymptotic Riemann-Roch)

If $\deg(\mathcal{L}) >> 0$ *then* $\dim H^0(Y,\mathcal{L}) \neq 0$.

Minkowski Theorem

Theorem (Minkowski)

Let $B \subset \mathbb{R}^N$ be a compact, convex subset symmetric with respect to the origin. Let Λ be a lattice of $\mathbb{R}^N$. If

$$\operatorname{Vol}(\mathbb{R}^N/\Lambda) \leq 2^{-N} \operatorname{Vol}(B),$$

then there exists an element $s \in B \cap \Lambda$, with $s \neq 0$.

What is the relationship between Minkowski Theorem and Riemann-Roch Theorem?

Hermitian line bundles

Let $X = \operatorname{Spec} \mathcal{O}_K$. A line bundle $\mathcal{L}$ over X is a rank one projective module over $\mathcal{O}_K$.

How we can extend $\mathcal{L}$ to $\overline{X} = X \cup \Sigma$?

What we need is a device that tells us when a rational section of $\mathcal{L}$ has a zero or a pole at a point of Σ.

For every $\sigma \in \Sigma$ we put a Hermitian metric, $\| \cdot \|_\sigma$, on the vector space $\mathcal{L}_\sigma = \mathcal{L} \underset{\sigma}{\otimes} \mathbb{C}$.

The space $\bigoplus \mathcal{L}_\sigma$ has a canonical antilinear involution, F_∞, that leaves $\mathcal{L}$ invariant. We assume that the above set of metrics is invariant under this involution.

We observe that $(\bigoplus \mathcal{L}_\sigma)^{F_\infty} \cong \mathbb{R}^{[K:\mathbb{Q}]}$, and the above metrics induce a norm on this space.

Global sections

We write $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_\sigma\}_\sigma)$.

Definition

Given a rational section $s \in \mathcal{L} \otimes K$ and a complex immersion σ of K we say that s is regular on σ if $\|s\|_\sigma \leq 1$. We say that s has a pole on σ if $\|s\|_\sigma > 1$.
Therefore we write

$$H^0(\overline{X}, \overline{\mathcal{L}}) = \{s \in \mathcal{L} \mid \|s\|_\sigma \leq 1,\ \forall \sigma \in \Sigma\}.$$

Therefore "global sections" are "small sections".

The arithmetic degree

The degree of a line bundle counts the number of zeros of a rational section minus the number of poles. This number is well defined thanks to the residue formula. This leads to the following definition of arithmetic degree.

Definition

Let s be any section of $\mathcal{L}$. Then we define

$$\widehat{\deg}(\overline{\mathcal{L}}) = \log(\#(\mathcal{L}/(\mathcal{O}_K \cdot s))) - \sum_{\sigma \in \Sigma} \frac{1}{e_\sigma} \log \|s\|_\sigma,$$

where $e_\sigma = 1$ if σ is real and $e_\sigma = 2$ otherwise.
This number is well defined as a consequence of the product formula.

The arithmetic asymptotic Riemann-Roch Theorem

The line bundle $\mathcal{L}$, defines a lattice in the vector space $(\bigoplus \mathcal{L}_\sigma)^{F_\infty} \cong \mathbb{R}^{[K:\mathbb{Q}]}$. Recall that this vector space has a norm. Then

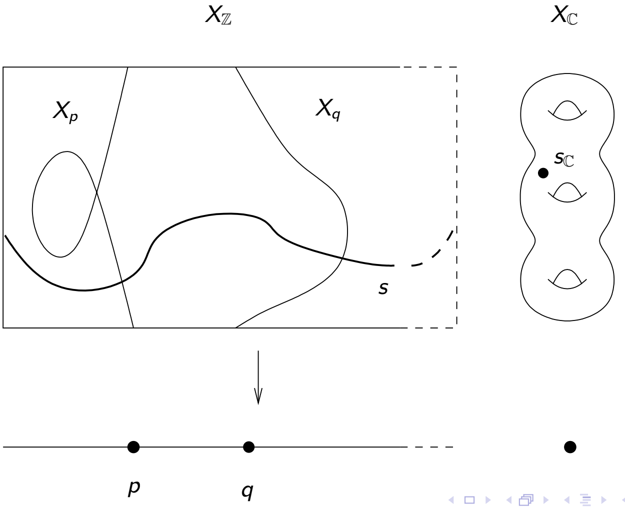
$$\widehat{\deg}(\overline{\mathcal{L}}) = -\log \mathrm{Vol}(\mathbb{R}^{[K:\mathbb{Q}]}/\mathcal{L}) + \frac{1}{2}\log |D_K|.$$

Therefore, Minkowski Theorem implies

Theorem (Arithmetic asymptotic Riemann-Roch Theorem)

If $\widehat{\deg}(\overline{\mathcal{L}}) >> 0$, *then* $H^0(\overline{X}, \overline{\mathcal{L}}) \neq 0$.

Arithmetic variety



Truncated relative cohomology groups

Relative cohomology

Let $f : A^* \longrightarrow B^*$ be a morphism of complexes of abelian groups.

Definition

The simple complex associated to f is the complex

$$s(f)^n = A^n \oplus B^{n-1}, \qquad \mathrm{d}(a,b) = (\mathrm{d}\, a, f(a) - \mathrm{d}\, b).$$

The relative cohomology groups of f are

$$H^*(A,B) = H^*(s(f)).$$

A long exact sequence

Recall that, for a complex of abelian groups A^*, the k-th shift is defined as

$$A[k]^n = A^{k+n}, \qquad \mathrm{d} = (-1)^k \,\mathrm{d}\,.$$

Let $f : A^* \longrightarrow B^*$ as before. There are natural morphisms

$$\begin{array}{rcl} \omega : s(f) & \longrightarrow & A \\ (a,b) & \longmapsto & a \end{array} \qquad \begin{array}{rcl} \mathrm{b} : B[1] & \longrightarrow & s(f) \\ b & \longmapsto & (0,-b) \end{array}$$

and a short exact sequence

$$0 \longrightarrow B[-1] \xrightarrow{\mathrm{b}} s(f) \xrightarrow{\omega} A \longrightarrow 0$$

That induces a long exact sequence

$$\dots \longrightarrow H^n(A,B) \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow \dots$$

The simple, the kernel and the co-kernel

The simple of a morphism of complexes is a generalization of the kernel of a monomorphism and the cokernel of an epimorphism.

Lemma

Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of abelian groups. Then there are natural quasi-isomorphisms

$$\begin{array}{ccc} s(f) & \longrightarrow & C[-1] \\ (a,b) & \longmapsto & g(b) \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & s(g) \\ a & \longmapsto & (f(a),0). \end{array}$$

Example: deRham cohomology with supports

Let M be a differentiable manifold, Y a closed subset of M and $U = M \setminus Y$. Let $A^*(M)$ denote the complex of real valued differential forms on M.
There is a restriction morphism $\rho : A^*(M) \longrightarrow A^*(U)$. By abuse of notation, if $\omega \in A^*(M)$ we will sometimes denote also by ω the restriction $\rho(\omega)$.

Definition

The deRham cohomology of M with support on Y is defined as

$$H^n_Y(M, \mathbb{R}) = H^n(s(\rho))$$

By definition there is a long exact sequence

$$\dots \longrightarrow H^n_Y(M, \mathbb{R}) \longrightarrow H^n(M, \mathbb{R}) \longrightarrow H^n(U, \mathbb{R}) \longrightarrow \dots$$

The product in cohomology with support I

The exterior product of differential forms induces a product in cohomology

$$H^n(M,\mathbb{R}) \otimes H^m(M,\mathbb{R}) \longrightarrow H^{n+m}(M,\mathbb{R})$$

That is graded commutative and associative.
By sheaf theory we know that, if Y and Z are closed subsets of M then there is a product

$$H^n_Y(M,\mathbb{R}) \otimes H^m_Z(M,\mathbb{R}) \longrightarrow H^{n+m}_{Y\cap Z}(M,\mathbb{R}).$$

How we can obtain such product with differential forms?

The product in cohomology with support II

First observe that if we write $U = M \setminus Y$ and $V = M \setminus Z$, then there is a short exact sequence

$$0 \longrightarrow A^*(U \cup V) \xrightarrow{u} A^*(U) \oplus A^*(V) \xrightarrow{v} A^*(U \cap V) \longrightarrow 0,$$

with $u(\omega) = (\omega, \omega)$ and $v(\omega, \eta) = \eta - \omega$. This exact sequence reflects the Mayer-Vietoris sequence in cohomology.
Therefore there is a quasi-isomorphism

$$A^*(U \cup V) \longrightarrow s(v)$$

There is also a well defined morphism $j : A^*(M) \longrightarrow s(v)$ given by $j(\omega) = ((\omega, \omega), 0)$.
We obtain an isomorphism

$$H^*_{Y \cap Z}(M, \mathbb{R}) = H^*(A^*(M), A^*(U \cup V)) \longrightarrow H^*(s(j))).$$

The product in cohomology with support III

There is a well defined morphism of complexes

$$s(A^*(M) \to A^*(U)) \otimes s(A^*(M) \to A^*(V)) \xrightarrow{\mu} s(j),$$

given, for (ω_1, η_1) of degree n and (ω_2, η_2) of degree m, by

$$\mu((\omega_1, \eta_1) \otimes (\omega_2, \eta_2)) = (\omega_1 \wedge \omega_2, ((\eta_1 \wedge \omega_2, (-1)^n \omega_1 \wedge \eta_2), (-1)^{n-1} \eta_1 \wedge \eta_2)).$$

Proposition

The above product induces the cup product in cohomology with support.

Truncated relative cohomology classes

Let $f : A^* \longrightarrow B^*$ be a morphism of complexes. Let σ denote the bête filtration:

$$\sigma^n A^m = \begin{cases} A^m, & \text{if } m \geq n, \\ 0, & \text{if } m < n. \end{cases}$$

Definition

The truncated relative cohomology groups of f are defined as

$$\widehat{H}^n(A, B) = H^n(\sigma^n A, B).$$

As we will see, the truncated cohomology groups are something between a cycle in the simple of f and a class in relative cohomology.

An explicit description.

Notation

Given a complex A we will denote

$$\mathrm{Z}\,A^n = \mathrm{Ker}(\mathrm{d} : A^n \longrightarrow A^{n+1}),$$
$$\widetilde{A}^n = A^n / \mathrm{d}\,A^{n-1}.$$

Note that there is a well defined map

$$\mathrm{d} : \widetilde{A}^{n-1} \longrightarrow \mathrm{Z}\,A^n.$$

Then

$$\widehat{H}^n(A, B) = \{(\omega, \widetilde{g}) \in \mathrm{Z}\,A^n \oplus \widetilde{B}^{n-1} \mid \mathrm{d}\,\widetilde{g} = f(\omega)\}.$$

Properties of truncated relative cohomology groups

Proposition

There are maps

$$\mathrm{cl}: \widehat{H}^n(A,B) \longrightarrow H(A,B) \qquad (\omega,\widetilde{g}) \longmapsto [(\omega,g)]$$

$$\omega: \widehat{H}^n(A,B) \longrightarrow \mathrm{Z}\,A^n \qquad (\omega,\widetilde{g}) \longmapsto \omega.$$

$$\mathrm{a}: \widetilde{A}^{n-1} \longrightarrow \widehat{H}^n(A,B) \qquad \widetilde{a} \longmapsto [(-\mathrm{d}\,a, -\widetilde{f(a)})]$$

$$\mathrm{b}: H^{n-1}(B) \longrightarrow \widehat{H}^n(A,B) \qquad [b] \longmapsto (0,-\widetilde{b}).$$

The following sequence is exact

$$H^{n-1}(A,B) \longrightarrow \widetilde{A}^{n-1} \xrightarrow{\mathrm{a}} \widehat{H}^n(A,B) \longrightarrow H^n(A,B) \longrightarrow 0.$$

Change of complexes

The following sequence is also exact:

$$0 \longrightarrow H^{n-1}(B) \xrightarrow{\mathrm{b}} \widehat{H}^{n}(A,B) \longrightarrow \mathrm{Z}\, A^{n} \longrightarrow H^{n}(B)$$

This means that the dependency on the complex A is much stronger than the dependency on the complex B. The following result will be important when defining products.

Lemma

If $g : B \longrightarrow C$ is a quasi-isomorphism, then the induced morphism

$$\widehat{H}^{n}(A,B) \longrightarrow \widehat{H}^{n}(A,C)$$

is an isomorphism.

Arithmetic Chow groups

Arithmetic varieties

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Let X be a regular projective flat scheme over $\mathcal{O}_K$.
Let Σ be the set of complex immersions of K. We write

$$X_\Sigma = \coprod_{\sigma \in \Sigma} X \underset{\sigma}{\times} \operatorname{Spec}(\mathbb{C}).$$

Then X_Σ has an antilinear involution F_∞ that defines a structure of real scheme. We write $X_\mathbb{R} = (X_\Sigma, F_\infty)$.
The real scheme $X_\mathbb{R}$ will play the role of the fibre at infinity of a compactification of X.
An arithmetic cycle will be a pair $(y, \mathfrak{g}_y)$, where y is an algebraic cycle on X and $\mathfrak{g}_y$ is an object on $X_\mathbb{R}$ related with y that we will construct using a cohomology theory.

A Gillet cohomology

Let $\mathcal{G}^*(*)$ be a graded complex of sheaves on the big Zariski site of regular schemes over $\mathbb{R}$ that satisfies Gillet axioms. This auxiliary cohomology will be the gluing that relates the geometry of X with a cohomology on $X_{\mathbb{R}}$.
The fact that $\mathcal{G}^*(*)$ satisfies Gillet axioms implies that, for any codimension p algebraic cycle $y_{\mathbb{R}}$ on $X_{\mathbb{R}}$ with support Y, there is a well defined class

$$\mathrm{cl}(y) \in H^{2p}_{Y}(X_{\mathbb{R}}, \mathcal{G}(p)).$$

Moreover if W is a subvariety of $X_{\mathbb{R}}$ of codimension $p-1$ and $f \in K^*(W)$ is a rational function with $y = \mathrm{div}(f)$, Y the support of y and $U = X_{\mathbb{R}} \setminus Y$ then there is a class

$$\mathrm{cl}(f) \in H^{2p-1}(U, \mathcal{G}(p)).$$

Compatibility of classes

Both classes are compatible in the sense that, if

$$\delta : H^{2p-1}(U, \mathcal{G}(p)) \longrightarrow H^{2p}_{Y}(X_{\mathbb{R}}, \mathcal{G}(p))$$

is the connection morphism then

$$\delta \operatorname{cl}(f) = \operatorname{cl}(\operatorname{div} f).$$

Arithmetic complexes I

A Gillet cohomology satisfies many properties. In many applications it is useful to use a complex with fewer properties. To this end we introduce the notion of arithmetic complexes.

Definition

Let $X_{\mathbb{R}}$ be a real scheme and $\mathcal{G}^*(*)$ a Gillet cohomology. An arithmetic $\mathcal{G}^*(*)$-complex is a graded complex of sheaves, $\mathcal{C}^*(*)$ in the Zariski topology of $X_{\mathbb{R}}$ provided with a structure morphism

$$\mathfrak{c} : \mathcal{G}^*(*) \longrightarrow \mathcal{C}^*(*),$$

such that all the sheaves $\mathcal{C}^n(p)|_U$ are acyclic for all $n, p \in \mathbb{Z}$ and U open subset of X.

The group of sections of $\mathcal{C}^n(p)$ over U will be denoted $\mathcal{C}^n(U, p)$.

Arithmetic complexes II

The acyclicity of the sheaves $\mathcal{C}^n(p)|_U$ is equivalent to the Mayer-Vietoris principle.

Mayer-Vietoris principle

For any pair of open sets U, V of $X_{\mathbb{R}}$ the sequence

$$0 \to \mathcal{C}^n(U \cup V, p) \to \mathcal{C}^n(U, p) \oplus \mathcal{C}^n(V, p) \to \mathcal{C}^n(U \cap V, p) \to 0$$

is exact.

Moreover the above acyclicity allows us to compute the hypercohomology of $\mathcal{C}$ by means of the complex of global sections. Therefore, for Y a closed subset of $X_{\mathbb{R}}$ with $U = X_{\mathbb{R}} \setminus Y$, we will use the notation

$$H^*_{\mathcal{C}}(U, p) = H^*(\mathcal{C}(U, p)), \ \ H^*_{\mathcal{C},Y}(X, p) = H^*(\mathcal{C}(X, p), \mathcal{C}(U, p)).$$

Classes for cycles and functions

The structure morphism $\mathfrak{c} : \mathcal{G} \longrightarrow \mathcal{C}$ induces morphisms

$$H^*(U, \mathcal{G}(p)) \longrightarrow H^*_{\mathcal{C}}(U, p),$$
$$H^*_Y(X, \mathcal{G}(p)) \longrightarrow H^*_{\mathcal{C},Y}(X, p).$$

Therefore, for y an algebraic cycle and f a rational function as before, we obtain compaticle classes

$$\mathrm{cl}(y) \in H^{2p}_{\mathcal{C},Y}(X, p),$$
$$\mathrm{cl}(f) \in H^{2p-1}_{\mathcal{C}}(U, p).$$

Green objects I

Let y be a codimension p algebraic cycle on X. Then it defines an algebraic cycle $y_{\mathbb{R}}$ on $X_{\mathbb{R}}$. Let Y be the support of y and let $U = X_{\mathbb{R}} \setminus Y$.

Definition

The space of Green objects for the cycle y is

$$GO(y) = \left\{ \mathfrak{g} \in \widehat{H}^{2p}(\mathcal{C}(X,p), \mathcal{C}(U,p)) \mid \mathrm{cl}(\mathfrak{g}) = \mathrm{cl}(y) \right\}$$
$$= \left\{ (\omega, \widetilde{g}) \in \mathrm{Z}\,\mathcal{C}^{2p}(X,p) \oplus \widetilde{\mathcal{C}}^{2p-1}(U,p) \mid [\omega, \widetilde{g}] = \mathrm{cl}(y) \right\}$$

If $\mathfrak{g}$ and $\mathfrak{g}'$ are two Green objects for the same cycle y then $\mathfrak{g} - \mathfrak{g}' = \mathrm{a}(\eta)$, for some $\eta \in \widetilde{\mathcal{C}}^{2p-1}(X,p)$.

Green objects II

The Green objects for different cycles live in different spaces. To glue together all these spaces we have take a limit. Let $\mathcal{Z}^p$ denote the set of codimension p closed subsets of $X_{\mathbb{R}}$. We write

$$\widehat{H}^{2p}_{\mathcal{C},\mathcal{Z}^p}(X,p) = \varinjlim_{Y\in\mathcal{Z}^p} \widehat{H}^{2p}_{\mathcal{C},Y}(X,p).$$

If $\mathcal{C}$ satisfies a purity property then the maps $GO(y) \longrightarrow \widehat{H}^{2p}_{\mathcal{C},\mathcal{Z}^p}(X,p)$ are injective. We write

$$GO^p(X) = \bigcup_{y \text{ of cod } p} GO(y).$$

If $\mathfrak{g}_y$ and $\mathfrak{g}_{y'}$ are Green objects for the cycles y and y' then $\mathfrak{g}_y + \mathfrak{g}_{y'}$ is a Green object for the cycle $y + y'$.

Green objects and rational functions

We denote by $X^{(p-1)}$ the set of irreducible subvarieties of codimension $p-1$ and we write

$$R_p^{p-1}(X) = \bigoplus_{W \in X^{(p-1)}} K^*(W).$$

The elements of this group are called K_1-chains.

Definition

Let $f \in R_p^{p-1}(X)$. Write $y = \operatorname{div} f$, Y the support of $y_{\mathbb{R}}$ and $U = X_{\mathbb{R}} \setminus Y$. Then the Green object associated to f is

$$\mathfrak{g}(f) = \mathrm{b}(\mathrm{cl}(f)) \in GO(\operatorname{div} f),$$

where $\mathrm{b} : H_{\mathcal{C}}^{2p-1}(U, p) \longrightarrow \widehat{H}^{2p}(\mathcal{C}(X, p), \mathcal{C}(U, p))$.

Abstract arithmetic Chow groups

Definition

With the notations as above we write

$$\widehat{Z}^p(X,\mathcal{C}) = \{(z,\mathfrak{g}) \in Z^p(X) \oplus GO^p(X) \mid \mathrm{cl}(z) = \mathrm{cl}(\mathfrak{g})\},$$
$$\widehat{\mathrm{Rat}}^p(X,\mathcal{C}) = \{(\operatorname{div} f, \mathfrak{g}(f)) \mid f \in R_p^{p-1}\},$$
$$\widehat{\mathrm{CH}}^p(X,\mathcal{C}) = \widehat{Z}^p(X,\mathcal{C}) / \widehat{\mathrm{Rat}}^p(X,\mathcal{C}).$$

There is a dictionary between properties of $\mathcal{C}$ and properties of $\widehat{\mathrm{CH}}^*(X,\mathcal{C})$.

Classical arithmetic Chow groups

Logarithmic singularities at infinity

We want to recover the arithmetic Chow groups of Gillet and Soulé from this abstract setting.

Let X be a projective complex manifold D a normal crossings divisor and $U = X \setminus D$. We have intoduced in the previous lecture the sheaf of differential forms on X with logarithmic singularities along D, $\mathscr{E}^*_X(\log D)$. We denote by $E^*_X(\log D)$ complex of global sections. This complex computes the cohomology of U with its Hodge filtration.

In order to have a complex that only depends on U and not on X we define

$$E^*_{\log}(U) = \lim_{(\overline{X},D)} E^*_X(\log D),$$

where $(\overline{X}, D)$ runs over all the compactifications of U with $D = \overline{X} \setminus U$ a normal crossing divisor.

Deligne-Beilinson cohomology as a Gillet cohomology

Since $E^*_{\log}(U)$ is a Dolbeault algebra, we can construct the associated Deligne complex and we denote

$$\mathcal{D}_{\log}(U,p) = \mathcal{D}(E_{\log}(U),p).$$

If $U_{\mathbb{R}} = (U_{\mathbb{C}}, F_\infty)$ is a smooth quasi-projective real variety, we denote also by F_∞ the involution on $\mathcal{D}_{\log}(U_{\mathbb{C}},p)$ that acts as F_∞ on the space and as complex conjugation on the coefficients. We denote

$$\mathcal{D}_{\log}(U_{\mathbb{R}},p) = \mathcal{D}_{\log}(U_{\mathbb{C}},p)^{F_\infty}.$$

Theorem

The assignment $U \longmapsto \mathcal{D}_{\log}(U_{\mathbb{R}},p)$ is a graded complex of sheaves in the big Zariski site of regular real schemes that satisfies Gillet axioms.

An arithmetic complex

Since the sheaf $\mathcal{D}_{\log}$ satisfies Gillet axioms we can take it as our Gillet complex $\mathcal{G}$. Since it also satisfies the Mayer-Vietoris principle it is also an arithmetic $\mathcal{D}_{\log}$-complex with the identity as structure morphism.

Let y be a codimension p algebraic cycle on X with support Y and write $U = X_{\mathbb{R}} \setminus Y$. Then a Green object for y in the complex $\mathcal{D}_{\log}$ is a pair

$$(\omega_y, \widetilde{g}_y) \in \mathrm{Z}\,\mathcal{D}^{2p}_{\log}(X_{\mathbb{R}}, p) \oplus \widetilde{\mathcal{D}}^{2p-1}_{\log}(U, p)$$

with $\mathrm{d}_{\mathcal{D}}\, g_y = \omega_y$.

These Green objects are called Green forms

Green forms

Unfolding the definition of the Deligne complex we obtain that

$$\omega_y \in \left(E_{\mathbb{C}}^{p,p}(X) \cap (2\pi i)^p E_{\mathbb{R}}^{2p}(X)\right)^{F_\infty}, \ \mathrm{d}\,\omega_y = 0,$$

$$\widetilde{g}_y \in \left(E_{\mathbb{C}}^{p-1,p-1}(X) \cap (2\pi i)^{p-1} E_{\mathbb{R}}^{2p-2}(X)\right)^{F_\infty} / (\operatorname{Im} \partial + \operatorname{Im} \bar{\partial})$$

These forms are related by $\omega_y = -2\partial\bar{\partial}\widetilde{g}_y$. Finally the last condition is that the class $[(\omega_y, g_y)] \in H^{2p}_{\mathcal{D},Y}(X_{\mathbb{R}}, \mathbb{R}(p))$ is the class of y.

If $f \in K^*(X)$ is a rational function then the Green form $\mathfrak{g}(f)$ is given explicitely by

$$\mathfrak{g}(f) = (0, -\frac{1}{2}\log(f\bar{f})).$$

$\mathcal{D}_{\log}$-Arithmetic Chow groups

Since $\mathcal{D}_{\log}$ is an arithmetic complex we can define the arithmetic Chow groups with coefficients in $\mathcal{D}_{\log}$ that we denote $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log})$.

Properties:

1. $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log}) \otimes \mathbb{Q}$ is a commutative and associative algebra.
2. If $f : X \longrightarrow Y$ is a morphism of arithmetic varieties then there is an inverse image morphism
$$f^* : \widehat{\mathrm{CH}}^*(Y, \mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log}).$$
3. If $f : X \longrightarrow Y$ is a morphism of arithmetic varieties of relative dimension e, such that $f_{\mathbb{R}} : X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$ is smooth then there is a direct image morphism
$$f_* : \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}^{*-e}(Y, \mathcal{D}_{\log}).$$

Exact sequences

Theorem

The following sequences are exact

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) \xrightarrow{a} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{\log}) \xrightarrow{\zeta} \mathrm{CH}^{p}(X) \longrightarrow 0,$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) \xrightarrow{a} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{\log}) \xrightarrow{(\zeta,-\omega)} \mathrm{CH}^{p}(X) \oplus \mathrm{Z}\mathcal{D}^{2p}_{\log}(X,p) \xrightarrow{\mathrm{cl}+h} H^{2p}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) \longrightarrow 0,$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) \xrightarrow{a} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{\log})_0 \xrightarrow{\zeta} \mathrm{CH}^{p}(X)_0 \longrightarrow 0.$$

The algebraic degree and the arithmetic degree

The arithmetic Chow of $\operatorname{Spec}\mathbb{Z}$ are

$$\widehat{\mathrm{CH}}^0(\operatorname{Spec}\mathbb{Z}) = \mathrm{CH}^0(\operatorname{Spec}\mathbb{Z}) = \mathbb{Z},$$
$$\widehat{\mathrm{CH}}^1(\operatorname{Spec}\mathbb{Z}) = H^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{R}, \mathbb{R}(1)) = \mathbb{R}.$$

If X is an arithmetic variety of relative dimension d, there is a unique map $\pi : X \longrightarrow \operatorname{Spec}\mathbb{Z}$. We write, for $x \in \widehat{\mathrm{CH}}^d(X, \mathcal{D}_{\log})$ and $y \in \widehat{\mathrm{CH}}^{d+1}(X, \mathcal{D}_{\log})$,

$$\deg(x) = \pi_*(x),$$
$$\widehat{\deg}(y) = \pi_*(y).$$

Currents.

Let X be a complex algebraic manifold of dimension d. The sheaf $\mathscr{D}_X^n$ of currents of degree n on X is defined as follows. For any open subset U of X, the group $\mathscr{D}_X^n(U)$ is the topological dual of the group of sections with compact support $\Gamma_c(U, \mathscr{E}_X^{2d-n})$. The differential $\mathrm{d} : \mathscr{D}_X^n \longrightarrow \mathscr{D}_X^{n+1}$ is defined by

$$\mathrm{d}\, T(\varphi) = (-1)^n T(\mathrm{d}\, \varphi).$$

The complex $\mathscr{D}$ is a Dolbeault complex.
There is a well defined morphism of complexes $\mathscr{E}_X^n \longrightarrow \mathscr{D}_X^n$ that to a form ω assigns the current $[\omega]$ given by

$$[\omega](\eta) \longmapsto \frac{1}{(2\pi i)^d} \int_X \eta \wedge \omega.$$

This morphism is a quasi-isomorphism.

Examples of currents

Example

If ω is a locally integrable differential form, then there is an associated current $[\omega]$ given also by

$$[\omega](\eta) \longmapsto \frac{1}{(2\pi i)^d} \int_X \eta \wedge \omega.$$

In general $\mathrm{d}[\omega] \neq [\mathrm{d}\,\omega]$. The difference is called the residue of ω. If Y is a subvariety of X of dimension e. Let $\widetilde{Y}$ be a resolution of singularities of Y, and $\imath : \widetilde{Y} \longrightarrow X$ the induced map. Then, the current *integration along* Y, denoted by δ_Y, is defined by

$$\delta_Y(\eta) = \frac{1}{(2\pi i)^e} \int_{\widetilde{Y}} \imath^* \eta.$$

Green currents

Using currents we can give a criterion for a pair $(\omega, \widetilde{g})$ to represent the class of an algebraic cycle y. (That is the original definition of Green current)

Theorem

Let X be a complex algebraic manifold, and y a p-codimensional cycle on X with support Y. Let (ω, g) be a cycle in

$$s^{2p}(\mathcal{D}_{\log}(X,p) \longrightarrow \mathcal{D}_{\log}(X \setminus Y, p)).$$

Then, the form g is locally integrable and the class of the cycle (ω, g) in $H^{2p}_{\mathcal{D},Y}(X, \mathbb{R}(p))$ is equal to the class of y, if and only if

$$-2\partial\bar{\partial}[g]_X = [\omega] - \delta_y.$$

Comparison of arithmetic Chow groups

As we have seen in the previous slide a Green form for a cycle defines a Green current for the same cycle.

Theorem

The assignment $[y,(\omega_y,\widetilde{g}_y)] \mapsto [y, 2(2\pi i)^{d-p+1}[g_y]_X]$ *induces an isomorphism*

$$\Psi : \widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}^p(X),$$

which is compatible with products, pull-backs and push-forwards.

Hermitian vector bundles

Hermitian vector bundles

We have developed an arithmetic intersection theory. The other main ingredient is to extend the notion of vector bundles to the arithmetic setting and to develop a theory of characteristic classes. Let X as before be a projective regular flat scheme over $\mathcal{O}_K$. Let E be a rank r locally free sheaf on X.
What extra structure we need to add to E over $X_{\mathbb{R}}$ to "compactify" it?

Definition

A Hermitian vector bundle is a locally free sheaf E over X together with a hermitian metric h on $E_{\mathbb{C}}$ that is invariant under F_∞. We denote $\overline{E} = (E, h)$.

Intuitively the hermitian metric tells us when a section of E is regular on the fibres at infinity.

Line bundles

Let $\overline{\mathcal{L}} = (\mathcal{L}, h)$ be a hermitian line bundle.
We can define the first Chern class of $\overline{\mathcal{L}}$ as follows.
Let s be a rational section of $\mathcal{L}$. Then we write

$$\widehat{\mathrm{c}}_1(\overline{\mathcal{L}}) = [(\operatorname{div} s, (-2\partial\bar{\partial}(-\frac{1}{2}\log h(s,s)), -\frac{1}{2}\log h(s,s)))] \in \widehat{\mathrm{CH}}^1(X, \mathcal{D}_{\log})$$

It is easy to see that this class is independent of the choice of s.

Theorem

The map $\widehat{\mathrm{c}}_1$ induces an isomorphism of groups

$$\left\{ \begin{matrix} \textit{Isometry classes of} \\ \textit{Hermitian line bundles} \end{matrix} \right\} \longrightarrow \widehat{\mathrm{CH}}^1(X, \mathcal{D}_{\log}).$$

Heights

The formalism of arithmetic Chow groups allow us to define heights. The height of a cycle is a measure of its arithmetic complexity and is the arithmetic analogue of the degree of a cycle. Let $\overline{\mathcal{L}}$ be a Hermitian vector bundle and let $z \in \mathrm{Z}^p(X)$ be a codimension p algebraic cycle. We choose a Green form $\mathfrak{g}_z = (\omega_z, g_z)$ for z and we write

$$h_{\overline{\mathcal{L}}}(z) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{d-p+1}.(z, \mathfrak{g}_z)) - \frac{1}{(2\pi i)^d} \int_{X_{\mathbb{C}}} c_1(\overline{\mathcal{L}})^{d-p+1} \wedge g_z$$

Theorem (Bost-Gillet-Soulé)

If $\mathcal{L}$ is ample then for any real number $A > 0$ the set of effective cycles z with $h_{\overline{\mathcal{L}}}(z) < A$ and $\deg_{\mathcal{L}}(z) < A$ is finite.

Arithmetic characteristic classes of vector bundles

Theorem

Let ϕ be a symmetric power series in r variables with rational coeficients. Then there is a unique way to attach to every Hermitian vector bundle $\overline{E} = (E, h)$ a characteristic class

$$\widehat{\phi}(\overline{E}) \in \widehat{\mathrm{CH}}^{*}(X, \mathcal{D}_{\mathrm{log}}) \otimes \mathbb{Q}$$

satisfying the following properties

Functoriality. *When $f : Y \longrightarrow X$ is a morphism of arithmetic varieties, then*

$$f^{*}(\widehat{\phi}(\overline{E})) = \widehat{\phi}(f^{*}\overline{E}).$$

Normalization. When $\overline{E} = \overline{L}_1 \oplus \cdots \oplus \overline{L}_n$ is a orthogonal direct sum of hermitian line bundles, then

$$\widehat{\phi}(\overline{E}) = \phi(\widehat{c}_1(\overline{L}_1), \ldots, \widehat{c}_1(\overline{L}_n)).$$

Twist by a line bundle. Let $\phi(T_1 + T, \ldots, T_n + T) = \sum_{i \geq 0} \phi_i(T_1, \ldots, T_n) T^i$.
Let $\overline{L}$ be a Hermitian line bundle. Then

$$\widehat{\phi}(\overline{E} \otimes \overline{L}) = \sum_i \widehat{\phi}_i(\overline{E}) \widehat{c}_1(\overline{L})^i.$$

Compatibility with characteristic forms.

$$\omega(\widehat{\phi}(\overline{E})) = \phi(E, h).$$

Compatibility with Bott-Chern forms

The above characteristic classes are compatible with the Bott-Chern forms in the following sense.
Let $\overline{\xi}$ be a short exact sequence of Hermitian vector bundles

$$0 \longrightarrow (E', h') \longrightarrow (E, h) \longrightarrow (E'', h'') \longrightarrow 0.$$

Then

$$\widehat{\phi}((E', h') \oplus (E'', h'')) - \widehat{\phi}((E, h)) = \mathrm{a}(\phi(\overline{\xi}))$$

Arithmetic K_0

We want to generalize the isomorphism between isometry class of line bundles to higher dimensional vector bundles.

Definition

$\widehat{K}_0(X)$ is the quotient of the abelian group of pairs $(\sum_i n_i \overline{E} + \eta)$, where the $\overline{E}_i$ are Hermitian vector bundle and $\eta \in \bigoplus_p \mathcal{D}^{2p-1}_{\log}(X,p)$, by the subgroup generated by elements of the form

$$\overline{E}' + \overline{E}'' - \overline{E} - \mathrm{ch}(\overline{\xi})$$

for every exact sequence $\overline{\xi}$

$$0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0.$$

The Chern character

There is a well defined morphism

$$\mathrm{ch} : \widehat{K}_0(X) \longrightarrow \bigoplus \widehat{\mathrm{CH}}^p(X) \otimes \mathbb{Q},$$

given by $\mathrm{ch}(\overline{E}, \omega) = \widehat{\mathrm{ch}}(\overline{E}) + a(\omega)$.
This morphism induces an isomorphism

$$\mathrm{ch} : \widehat{K}_0(X) \otimes \mathbb{Q} \longrightarrow \bigoplus \widehat{\mathrm{CH}}^p(X) \otimes \mathbb{Q},$$