Arithmetic characteristic classes of log-singular Hermitian vector bundles

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Historical notes

1. Arakelov geometry was introduced by Arakelov in 1974 in the case of arithmetic surfaces.
2. Faltings in 1984 proved the Riemann-Roch theorem and the Hodge index theorem for arithmetic surfaces.
3. Deligne in 1985 shows how to avoid the condition of harmonicity.
4. In 1990 Gillet and Soulé generalize Arakelov geometry to higher dimensions.
5. Many variants and generalizations by Zhang, Maillot, Bost, Moriwaki, Kühn ...
6. The abstract version presented in this course is joint work with Kühn and Kramer.

Hirzebruch-Zagier formula

The Hilbert modular surface

Let $p \equiv 1 \mod 4$ be a prime and let $\mathcal{O}_K$ be the ring of integers of $K = \mathbb{Q}(\sqrt{p})$.
Let $\mathfrak{H}$ be the upper half plane. Then $X = \mathfrak{H}^2 / SL_2(\mathcal{O}_K)$ is a non-compact complex surface with finitely many singularities.
This surface can be compactified adding h cusps, where h is the class number of K.
According to Baily-Borel this compactified variety is a normal projective variety over $\mathbb{C}$.

Hirzebruch-Zagier cycles

For any $m > 0$ let $\widetilde{T}(m)$ be the set of all points of $\mathfrak{H}^2$ that satisfy any of the equations

$$a\sqrt{p}z_1z_2 + \lambda z_2 + \lambda' z_1 + b\sqrt{p} = 0,$$

with $a, b \in \mathbb{Z}$, $\lambda \in \mathcal{O}_K$, $\lambda\lambda' + abp = m$.
The set $\widetilde{T}(m)$ is invariant under $SL_2(\mathcal{O}_K)$ and its image in X, denoted $T(m)$, has finitely many components. This cycle is called a Hirzebruch-Zagier cycle.
If m is the norm of an ideal in $\mathcal{O}_K$, then $T(m)$ is a non-compact divisor on X, birational to a linear combination of modular curves. In this case we say that $T(m)$ is *isotropic*. If m is not the norm of an ideal in $\mathcal{O}_K$, then $T(m)$ is a compact divisor on X, birational to a linear combination of Shimura curves. In that case we say that $T(m)$ is *anisotropic*.

Compactified Hirzebruch-Zagier divisors

Let $\widetilde{X}$ be the surface obtained adding the cusps to X and resolving the singularities of the cusps.
Let $T^c(m)$ denote the class in $CH^1(\widetilde{X})$ of the preimage of the adherence of $T(m)$ on X.
Let $\mathcal{M}_k$ denote the line bundle of modular forms of weight k. We denote by $T^c(0)$ the homology class defined by Poincaré duality by $-\frac{1}{2k}c_1(\mathcal{M}_k)$ for k sufficiently large.

Hirzebruch-Zagier Formula

Theorem (Hirzebruch-Zagier, Borcherds)

For any class K in $CH^1(\widetilde{X})$ the function

$$\Phi_K(z) = \sum_{m=0}^{\infty} (T^c(m) \cdot K) q^m$$

is a modular form of weight 2, level p and character χ_p.

Generalization

Let V be a $\mathbb{Q}$ vector space provided with an inner product of signature $(n, 2)$. $G = GSpin(V)$.
Let $B = \{w \in V(\mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) < 0\}/\mathbb{C}^* \subset \mathbb{P}(V(\mathbb{C}))$.
Then B is a Hermitian symmetric domain of dimension n.
To these data one can associate Shimura varieties M_K.
Depending on the dimension one recover modular and Shimura curves, products of these curves, Hilbert modular surfaces, Siegel modular 3-folds ...
The Shimura varieties as above have many special subvarieties that are the analogue of Hirzebruch-Zagier divisors.
By means of cohomology classes of special cycles of codimension r, Kudla and Millson have constructed Siegel modular forms of genus r and weight $\frac{n}{2} + 1$.

Arakelov analogues

1. Gross-Zagier formula.
2. Computations of Kudla-Rapoport-Yang on special cycles of arithmetic varieties associated to $O(n,2)$.
3. Computations of Bost and Kühn on modular curves.
4. Conjectures of Kramer, Köhler and Maillot-Roessler on the arithmetic Hodge numbers of a semi-abelian fibration.

There are many technical problems.

1) Problems related with integral models.

2) In general Shimura varieties are non compact.

Singular metrics on Shimura varieties

Logarithmic line bundles

Observation

The natural metrics that appear in the line bundles on a compactification of the moduli space of abelian varieties have logarithmic singularities.

Lemma (Faltings)

Let $X \subset \mathbb{P}^n_{\mathbb{Z}}$ be a Zariski closed subset, let $Y \subset X$ be closed.
Let $\| \ \|$ be a Hermitian metric on $\mathcal{O}(1)|_{X(\mathbb{C})\setminus Y(\mathbb{C})}$, with logarithmic singularities along Y.
Let K be a number field. Let h be the height associated to $\| \ \|$.
$c > 0$.
Then

$$\{x \in X(K) \setminus Y(K) \mid h(x) \leq c\}$$

is finite.

Poincaré metric

Let $\overline{X}$ be a complex manifold of dimension n, and let $X = \overline{X} \setminus D$, with D a normal crossing divisor.
Let U be an open coordinate set, with $U \setminus D \cong (\Delta_\epsilon^*)^k \times \Delta_\epsilon^{n-k}$, and ϵ small.

The Poincaré metric on Δ_ϵ^* is $ds^2 = \frac{|dz|^2}{|z|^2(\log|z|)^2}$
The standard metric in Δ_ϵ is $ds^2 = |dz|^2$.

Let ω_U be the product metric in $U \setminus D$.

Poincaré growth and Good forms

Definition

A complex valued p-form, η, has *Poincaré growth* if, for any open coordinate set U as above, one has the estimate

$$|\eta(t_1, \dots, t_p)|^2 \leq C\omega_U(t_1, t_1) \dots \omega_U(t_p, t_p).$$

for all tangent vectors $t_1, \dots, t_p$ in $U \setminus D$.
A p-form η is *good* if η and $d\eta$ have Poincaré growth.

Theorem (Mumford)

1. *A good form is locally L^1.*
2. *Let $[\eta]$ be the associated current. Then $d[\eta] = [d\eta]$.*

Good metrics

Definition

Let E be a vector bundle on $\overline{X}$ and let h be a Hermitian metric in $E|_X$. We say h is *good* if, for all open coordinate sets as above and local frames of E
i) $|h_{ij}|, (\det h)^{-1} \leq C(\sum_{i=1}^{k} \log |z_i|)^N,\ C > 0.$
ii) The 1-form $(\partial h.h^{-1})_{ij}$ are good.

Theorem (Mumford)

If h is a good metric of E, then for all k, the k-th Chern form $c_k(E,h)$ is a good form and the current $[c_k(E,h)]$ represents the k-th Chern class of E.

Conclusion

The logarithmic line bundles and the good Hermitian vector bundles behave in many situations like smooth Hermitian vector bundles.

Fully decomposed automorphic bundles

Let $B = K\backslash G$ be a Hermitian symmetric domain.
Inside the complexification $G_{\mathbb{C}}$ of G, there is a parabolic subgroup of the form $P_+ \cdot K_{\mathbb{C}}$ and an equivariant immersion

$$B \subset \check{B} = G_{\mathbb{C}}/P_+ \cdot K_{\mathbb{C}},$$

that induces a complex structure on B.
Let $\sigma : K \longrightarrow GL(n, \mathbb{C})$ be a representation of K. Then σ defines a G-equivariant vector bundle E_0 on B.
We complexify σ and we extend it trivially to $P_+ \cdot K_{\mathbb{C}}$ by letting it kill P_+. Then σ defines a holomorphic $G_{\mathbb{C}}$-equivariant vector bundle $\check{E}_0$ on $\check{B}$ with $E_0 = \iota^*(\check{E}_0)$. This induces a holomorphic structure on E_0.

Fully decomposed automorphic bundles

Let Γ be a neat arithmetic group acting on B. Then $X = \Gamma \backslash B$ is a smooth quasi-projective complex variety, and E_0 defines a holomorphic vector bundle E on X. The vector bundles obtained in this way (with σ extended trivially) will be called *fully decomposed automorphic vector bundles*.

Let h_0 be a G-equivariant Hermitian metric on E_0. Such metrics exist by the compacity of K. Then h_0 determines a Hermitian metric h on E_0. Let $\overline{X}$ be a smooth toroidal compactification of X with $D = \overline{X} \setminus X$ a normal crossing divisor.

Fully decomposed automorphic bundles

Theorem (Mumford)

There exists a unique extension of E_0 to a vector bundle E on $\overline{X}$ such that the Hermitian metric h is good along D.

Warning

The result is not true for non fully decomposed automorphic vector bundles. i.e. where the extension of the representation σ to $P_+ \cdot K_{\mathbb{C}}$ is not trivial on P_+. Similarly, the Hodge metric associated to a variation of polarized Hodge structures is not in general good.

Objective

Objective:
To extend the formalism of Arakelov geometry to cover fully decomposed automorphic vector bundles.

Log and log-log forms

Examples of singular forms

Let $\overline{\mathcal{L}}$ be a Hermitian line bundle, log singular along $z=0$, and let s be a non-vanishing regular section. Assume that in a neighborhood of $z=0$,

$$h(s)=C(z)(\log(1/|z|))^N,$$

with C smooth and non zero. Then the associated Green function will satisfy

$$\log h(s)=N\log\log(1/|z|)+\phi(z).$$

If we take the derivative we obtain Poincaré like singularities:

$$\partial\log h(s)=N\frac{-\mathrm{d}\,z}{z\log 1/|z|}+\partial\phi(z).$$

Local coordinates

Recall that, to represent classes in cohomology with support we used a complex of differential forms with logarithmic singularities. When mixing log-singularities with log-log singularities it is convenient to use growth conditions.
Let X be a smooth complex variety of dimension n and $D \subset X$ a normal crossing divisor. Write $V = X \setminus D$ and let $j : V \longrightarrow X$ be the inclusion.
Let Δ be an open coordinate subset with coordinates $(z_1, \dots, z_n)$, and let $z_1 \dots z_k = 0$ be a local equation for D. Put $r_i = \|z_i\|$.
We will always assume that the r_i are small enough.

Log growth functions

Definition

A function f has log growth along D if, for any coordinate subset as above, and all multi-indices α, β, the following estimate holds

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} f(z_1, \dots, z_d)\right| \leq C_{\alpha,\beta} \frac{\left|\prod_{i=1}^{k} \log(1/r_i)\right|^{N_{\alpha,\beta}}}{\left|z^{\alpha^{\leq k}} \bar{z}^{\beta^{\leq k}}\right|},$$

where

$$z^{\alpha^{\leq k}} = \prod_{i=1}^{k} z_i^{\alpha_i}.$$

Log growth forms

Definition

The sheaf of differential forms on X with log growth along D, denoted $\mathscr{E}^*_X \langle D \rangle$, is the subalgebra of $j_* \mathscr{E}^*_V$ generated locally by the log growth functions and the forms

$$\frac{d z_i}{z_i}, \ \frac{d \bar{z}_i}{\bar{z}_i}, \qquad \text{for } i = 1, \dots, k,$$
$$d z_i, \ d \bar{z}_i, \qquad \text{for } i = k+1, \dots, n.$$

For shorthand a log growth form will be called a log form.

Properties of the sheaf of log forms

Properties of log forms

1. The sheaf $\mathscr{E}_X^* \langle D \rangle$ is closed under ∂, $\bar{\partial}$, $\wedge$ and complex conjugation. Therefore it has a structure of Dolbeault algebra.
2. It is stable under inverse images.
3. If $\Omega^*(\log D)$ denotes the sheaf of holomorphic forms with logarithmic poles along D, then the natural inclusion

$$\Omega^*(\log D) \longrightarrow \mathscr{E}_X^* \langle D \rangle$$

is a filtered quasi-isomorphism with respect to the Hodge filtration.

Log-log growth functions

Definition

A function f has log-log growth along D if, for any coordinate subset as above, and all multi-indices α, β, the following estimate holds near D

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} f(z_1,\dots,z_d)\right| \leq C_{\alpha,\beta}\frac{\left|\prod_{i=1}^{k}\log(\log(1/r_i))\right|^{N_{\alpha,\beta}}}{\left|z^{\alpha^{\leq k}}\bar{z}^{\beta^{\leq k}}\right|}.$$

Log-log growth forms

Definition

The sheaf of differential forms on X with log-log growth along D, is the subalgebra of $j_*\mathscr{E}^*_V$ generated locally by the log-log functions and the forms

$$
\begin{aligned}
&\frac{dz_i}{z_i\log(1/r_i)},\ \frac{d\bar{z}_i}{\bar{z}_i\log(1/r_i)}, && \text{for } i=1,\dots,k,\\
&dz_i,\ d\bar{z}_i, && \text{for } i=k+1,\dots,n.
\end{aligned}
$$

Log-log forms

Warning: The sheaf of log-log growth forms is not closed under ∂ and $\bar{\partial}$.

Definition

We say that a complex differential form ω is *log-log along D* if the differential forms ω, $\partial\omega$, $\bar{\partial}\omega$ and $\partial\bar{\partial}\omega$ have log-log growth along D. The sheaf of differential forms log-log along D will be denoted by $\mathscr{E}^*_X\langle\langle D\rangle\rangle$.

Properties of log-log forms

Properties of log-log forms

1. The sheaf $\mathscr{E}_X^* \langle\langle D \rangle\rangle$ is closed under ∂, $\bar{\partial}$, $\wedge$ and complex conjugation. Therefore it has a structure of Dolbeault algebra.
2. The sections of $\mathscr{E}_X^* \langle\langle D \rangle\rangle$ are locally integrable with zero residue.
3. It is stable under inverse images.
4. If Ω^* denotes the sheaf of holomorphic forms on X, then the natural inclusion
$$\Omega^* \longrightarrow \mathscr{E}_X^* \langle\langle D \rangle\rangle$$
is a filtered quasi-isomorphism with respect to the Hodge filtration.

Log-log singularities are so mild that they do not change the cohomology

Mixed growth

We can mix together log and log-log forms.
Let D_1 and D_2 be two normal crossing divisors on X such that $D_1 \cup D_2$ is also a normal crossing divisor.
We can define the sheaf $\mathscr{E}^*_X \langle D_1 \langle D_2 \rangle\rangle$ of differential forms that are log along D_1 and log-log along D_2.

Theorem

The natural inclusion

$$\Omega^*_X(\log D_1) \longrightarrow \mathscr{E}^*_X \langle D_1 \langle D_2 \rangle\rangle$$

is a filtered quasi-isomorphism with respect to the Hodge filtration.

Real Deligne-Beilinson cohomology

Let U be a non proper smooth complex algebraic variety, $D \subset U$ a normal crossing divisor.
Write

$$\mathscr{E}^*_{\log} \langle\langle D \rangle\rangle (U) = \varinjlim_{(\overline{U}, D')} \mathscr{E}^*_X \langle D' \langle \overline{D} \rangle\rangle ,$$

where the limit is taken over all compactifications $(\overline{U}, D')$ of U, such that $D' \cup \overline{D}$ is a normal crossings divisor.
The space of global sections $E^*_{\log} \langle\langle D \rangle\rangle (U)$ is a Dolbeault algebra.

Corollary

The complex $\mathcal{D}^*(E_{\log} \langle\langle D \rangle\rangle (U), p)$ computes the Deligne-Beilinson cohomology of U.

Real varieties

Let $U_{\mathbb{R}}$ be a real variety and assume that $D_{\mathbb{R}}$ is defined over $\mathbb{R}$. Then there is an involution F_∞ of $\mathcal{D}^*(E_{\log}\langle\langle D_{\mathbb{C}}\rangle\rangle(U_{\mathbb{C}}), p)$ that acts as complex conjugation on the space and the coefficients. Then we write

$$\mathcal{D}^*(E_{\log}\langle\langle D_{\mathbb{R}}\rangle\rangle(U_{\mathbb{R}}), p) = \mathcal{D}^*(E_{\log}\langle\langle D_{\mathbb{C}}\rangle\rangle(U_{\mathbb{C}}), p)^{F_\infty}$$

Corollary

The complex $\mathcal{D}^*(E_{\log}\langle\langle D_{\mathbb{R}}\rangle\rangle(U_{\mathbb{R}}), p)$ computes the real Deligne-Beilinson cohomology of the real variety $U_{\mathbb{R}}$.

Log-log arithmetic Chow groups

Log-log Arithmetic Chow groups

Let X be an arithmetic variety over $\mathcal{O}_K$, of relative dimension d.
Let D be a fixed normal crossing divisor of $X_{\mathbb{R}}$.
Usually, X will be an integral model over $\mathcal{O}_K$ (or a localization of it) of a toroidal compactification of a Shimura variety and D will be the boundary divisor.
The assignment

$$U_{\mathbb{R}} \longmapsto \mathcal{D}^n(E_{\log}\langle\langle D\rangle\rangle(U_{\mathbb{R}}), p) =: \mathcal{D}^n_{\langle\langle D\rangle\rangle}(U, p)$$

is an arithmetic $\mathcal{D}_{\log}$-complex.
Therefore applying the abstract machinery of the previous talk, we define the arithmetic Chow groups with values in the complex of log-log forms, that we will denote

$$\widehat{\mathrm{CH}}^*(X, \langle\langle D\rangle\rangle).$$

Properties

Theorem

1. $\widehat{\mathrm{CH}}^*(X, \langle\langle D \rangle\rangle) \otimes \mathbb{Q}$ *is a commutative and associative ring.*
2. *If* D *and* E *are normal crossing divisors on* $X_{\mathbb{R}}$, $Y_{\mathbb{R}}$ *resp. and* $f : X \longrightarrow Y$ *is a morphism such that* $f^{-1}(E) \subset D$, *then there is a morphism*

$$f^* : \widehat{\mathrm{CH}}^*(Y, \langle\langle E \rangle\rangle) \longrightarrow \widehat{\mathrm{CH}}^*(X, \langle\langle D \rangle\rangle).$$

3. *If* X *is proper over* $\mathrm{Spec}(\mathcal{O}_K)$, *of relative dimension* n, *there is a well defined map*

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{n+1}(X, \langle\langle D \rangle\rangle) \xrightarrow{\pi_*} \widehat{\mathrm{CH}}^1(\mathcal{O}_K) \xrightarrow{\widehat{\deg}} \mathbb{R}.$$

Exact sequences

Theorem

The following sequences are exact

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\langle\langle D\rangle\rangle}(X,p) \xrightarrow{a} \widehat{\mathrm{CH}}^{p}(X,\langle\langle D\rangle\rangle) \xrightarrow{\zeta} \mathrm{CH}^{p}(X) \longrightarrow 0,$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X,p) \xrightarrow{a} \widehat{\mathrm{CH}}^{p}(X,\langle\langle D\rangle\rangle) \xrightarrow{(\zeta,-\omega)} \mathrm{CH}^{p}(X) \oplus Z\mathcal{D}^{2p}_{\langle\langle D\rangle\rangle}(X,p) \xrightarrow{\mathrm{cl}+h} H^{2p}_{\mathcal{D}}(X,p) \longrightarrow 0,$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X,p) \xrightarrow{a} \widehat{\mathrm{CH}}^{p}(X,\langle\langle D\rangle\rangle)_0 \xrightarrow{\zeta} \mathrm{CH}^{p}(X)_0 \longrightarrow 0.$$

Green forms

Let X be a complex projective manifold, y a codimension p cycle, with support Y and $U = X \setminus Y$.
Thanks to the fact that we have precise control on the cohomology of the complex of log-log forms, we can give a precise criterion for Green forms.

Theorem

Let (ω, g) be a cycle in

$$s^{2p}(\mathcal{D}_{\langle\langle D\rangle\rangle}(X, p) \longrightarrow \mathcal{D}_{\langle\langle D\rangle\rangle}(U, p)).$$

Then, the form g is locally integrable and the class of the cycle (ω, g) in $H^{2p}_{\mathcal{D},Y}(X, \mathbb{R}(p))$ is equal to the class of y, if and only if

$$-2\partial\bar{\partial}[g]_X = [\omega] - \delta_y.$$

Products

The product in Deligne cohomology

Recall that if X is a complex projective variety. Deligne cohomology of X can be computed as the cohomology of the simple complex

$$E_{\mathbb{R}}(X,p)_{\mathcal{D}} := s((2\pi i)^p E_{\mathbb{R}^*}(X) \oplus F^p E^*_{\mathbb{C}}(X) \xrightarrow{u} E^*_{\mathbb{C}}(X)).$$

This complex has a family of products. For every $\alpha \in [0,1]$, let

$$\begin{aligned}(r_p, f_p, \omega_p) \cup_\alpha (r_q, f_q, \omega_q) = \\ (r_p \wedge r_q, f_p \wedge f_q, \alpha(\omega_p \wedge r_q + (-1)^n f_p \wedge \omega_q) \\ +(1-\alpha)(\omega_p \wedge f_q + (-1)^n r_p \wedge \omega_q)) .\end{aligned}$$

All these products are homotopically equivalent. For $\alpha = 0, 1$ the product is associative and for $\alpha = 1/2$ the product is graded commutative. Therefore it gives a well defined associative and commutative product in Deligne cohomology.

The product in the Deligne complex

There are explicit homotopy equivalences

$$\varphi : \mathcal{D}(E_{\mathbb{R}}(X), p) \longrightarrow E_{\mathbb{R}}(X, p)_{\mathcal{D}}, \qquad \psi : E_{\mathbb{R}}(X, p)_{\mathcal{D}} \longrightarrow \mathcal{D}(E_{\mathbb{R}}(X), p),$$

and we define, for $x, y \in \mathcal{D}(E_{\mathbb{R}}(X), p)$

$$x \bullet y = \psi(\varphi(x) \cup_{\alpha} \varphi(y)).$$

This product does not depend on α and it is graded-commutative and associative up to homotopy.

Example

If $x \in \mathcal{D}^{2p}(E_{\mathbb{R}}(X), p)$, then $x \bullet y = x \wedge y$.
If $x \in \mathcal{D}^{2p-1}(E_{\mathbb{R}}(X), p)$ and $y \in \mathcal{D}^{2q-1}(E_{\mathbb{R}}(X), q)$, then

$$x \bullet y = -\partial x \wedge y + \bar{\partial} x \wedge y + x \wedge \partial y - x \wedge \bar{\partial} y.$$

The product of Green objects

Let $\mathcal{D}$ denote one of the sheaves $\mathcal{D}_{\log}$ or $\mathcal{D}_{\langle\langle D\rangle\rangle}$.
Let y and z be two cycles that intersect properly and let Y and Z be their support. Let p, q be their respective codimensions and let $r = p + q$. Write $U = X \setminus Y$, $V = X \setminus Z$.
Let $\mathfrak{g}_y = (\omega_y, \widetilde{g}_y)$ and $\mathfrak{g}_z = (\omega_z, \widetilde{g}_z)$ be Green forms for the cycles y and z respectively.
Guided by the product in cohomology with support we put

$$\mathfrak{g}_y * \mathfrak{g}_z = \left(\omega_y \bullet \omega_z, ((g_y \bullet \omega_z, \omega_y \bullet g_z), -g_y \bullet g_z)^{\sim}\right),$$

which is an element of
$\widehat{H}^{2r}(\mathcal{D}(X, r), s(\mathcal{D}(U, r) \oplus \mathcal{D}(V, r) \to \mathcal{D}(U \cap V, r)).$

The product of Green objects

By the Mayer Vietoris property, we know that

$$\widehat{H}^{2r}(\mathcal{D}(X,r), s(\mathcal{D}(U,r) \oplus \mathcal{D}(V,r) \to \mathcal{D}(U \cap V, r))) \cong \widehat{H}^{2r}(\mathcal{D}(X,r), \mathcal{D}(U \cup V, r)).$$

How can we make explicit this isomorphism?

The product of Green objects

Let $\widehat{X}$ be a resolution of singularities of $Y \cap Z$ such that the strict transforms of y and z do not meet.
Let $\sigma_{y,z}$ be a smooth function on $\widehat{X}$ that has the value 1 in a neighborhood of the strict transform of y and 0 in a neighborhood of the strict transform of z. Put

$$\sigma_{z,y} = 1 - \sigma_{y,z} \qquad \text{and}$$

$$g_y * g_z = \sigma_{z,y} g_y \wedge (-2\partial\bar{\partial} g_z) + (-2\partial\bar{\partial}\sigma_{y,z} g_y) \wedge g_z.$$

Proposition

Then the pair $(\omega_y \wedge \omega_z, (g_y * g_z)^{\sim})$ represents the product $\mathfrak{g}_y * \mathfrak{g}_z$.

The product of Green objects

Assume that we are in the case $\mathcal{D} = \mathcal{D}_{\log}$, and that $p + q = d + 1$. Then using Stokes

$$\int_X g_y * g_z = \int_X g_y \wedge \omega_z + (4\pi i)\, \mathrm{d}\left(\mathrm{d^c}(\sigma_{YZ} g_y) \wedge g_z - \sigma_{YZ} g_y \wedge \mathrm{d^c}\, g_z\right) = \int_X (g_y \wedge \omega_z + \delta_y \wedge g_z).$$

This is the classical Gillet-Soulé star product.
Note that the last step is not justified when we are in the case $\mathcal{D}_{\langle\langle D \rangle\rangle}$ and $Y \subset D$.

The star product on modular curves

Let X be a complex projective curve. $S \subset X$ a finite set of points. $\overline{\mathcal{L}}$ a good hermitian vector bundle on X such that, near a point $s_i \in S$, a non vanishing regular section of $\mathcal{L}$ satisfies

$$\|l\| = (\log(|t|))^{\alpha_i}\varphi(t),$$

where t is a local coordinate and φ is a positive continuous function such that

$$\frac{\partial\varphi(t)}{\partial t} \leq \frac{\beta}{|t|^{1-\rho}}, \quad \frac{\partial\varphi(t)}{\partial\bar{t}} \leq \frac{\beta}{|t|^{1-\rho}}, \quad \frac{\partial^2\varphi(t)}{\partial t\partial\bar{t}} \leq \frac{\beta}{|t|^{2-\rho}}.$$

To every section l of $\mathcal{L}$ we can associate the Green form

$$\mathfrak{g}_l = (-2\partial\bar{\partial}(-\frac{1}{2}\log\|l\|^2), -\frac{1}{2}\log\|l\|^2) = (\omega_l, g_l).$$

The star product on modular curves

If l and m are sections of $\mathcal{L}$ whose divisor intersects S, then the formula

$$\int_X g_l * g_m = \int_X (g_l \wedge \omega_m + \delta_l \wedge g_m)$$

does not make sense because both terms diverge.

Nevertheless, using Stokes theorem and the general formula for the product, one can derive Kühn's formula for the star product

$$\int_X g_l * g_m = \int_X (g_l \wedge \omega_m + \omega_l \wedge g_m + (4\pi i)\, \mathrm{d}\, g_l \wedge \mathrm{d}^c\, g_m) + (2\pi i) \sum_{s_i \in S} (\mathrm{ord}_{s_i}(l) + \mathrm{ord}_{s_i}(m))\alpha_i$$

Log singular Hermitian metrics

Log-singular Hermitian metrics

Let X be a complex manifold, D a normal crossing divisor and $U = X \setminus D$. Let E be a rank n vector bundle on X and let E_0 be the restriction of E to U.

Definition

A smooth metric on E_0 is said to be *log-singular* along D if for every $x \in D$, there exist a trivializing open coordinate neighborhood V adapted to D with holomorphic frame $\xi = \{e_1, \dots, e_n\}$, such that, writing $h(\xi)_{ij} = h(e_i, e_j)$, then

1. The functions $h(\xi)_{ij}$, $\det(h(\xi))^{-1}$ belong to $\Gamma(V, \mathscr{E}^0_X \langle D \rangle)$,
2. The 1-forms $(\partial h(\xi).h(\xi)^{-1})_{ij}$ belong to $\Gamma(V, \mathscr{E}^{1,0}_X \langle\langle D \rangle\rangle)$.

A vector bundle provided with a log-singular Hermitian metric will be called a *log-singular Hermitian vector bundle*.

Properties

Let $\overline{E}$, $\overline{F}$ be Hermitian vector bundles on X, log-singular along D. $f : Y \longrightarrow X$ a holomorphic map and D' a normal crossing divisor on Y such that $f^{-1}(D) \subset D'$. Then

1. $f^{*}\overline{E}$ is log-singular along D'.
2. The tensor product $\overline{E} \otimes \overline{F}$, the exterior and symmetric powers $\Lambda^{n}\overline{E}$, $S^{n}\overline{E}$, the dual bundle $\overline{E}^{\vee}$ and the bundle of homomorphisms $\mathrm{Hom}(\overline{E}, \overline{F})$, with their induced metrics, are log-singular along D.
3. $\overline{E} \overset{\perp}{\oplus} \overline{F}$ is log-singular along D if and only if $\overline{E}$ and $\overline{F}$ are log-singular along D.

Warning. The concept of log-singular Hermitian metric is not closed under general sub-objects, quotients and extensions.

Bott-Chern forms of log-singular metrics

We want to define arithmetic characteristic classes from log-singular Hermitian vector bundles to the arithmetic Chow groups with values in $\mathcal{D}_{\langle\langle D\rangle\rangle}$.
The strategy is simple. One changes the log-singular metric by a smooth metric. Then we consider the Gillet-Soulé arithmetic characteristic class that lives in the theory with values in $\mathcal{D}_{\log}$.
Finally we correct the effect of the change of metric by using a Bott-Chern form. Therefore we are led to prove.

Proposition

Let E be a vector bundle on X, let h be a log-singular Hermitian metric on E and let h' be a smooth hermitian metric. Then the Bott-Chern forms for the change of metrics h, h' belongs to the space $E^*_X \langle\langle D\rangle\rangle$.

Iterated Bott-Chern forms

There is still the problem of the dependency on the chosen smooth Hermitian metric. To solve this problem we need to prove

Proposition

Let E be a vector bundle on X, let h be a log-singular Hermitian metric on E and let h' and h'' be two smooth Hermitian metrics. Then the iterated Bott-Chern form for the three metrics h, h', h'' belongs to the space $E^*_X \langle\langle D \rangle\rangle$.

As a consequence, the arithmetic characteristic classes obtained using the metric h' or the metric h'' differ by a boundary and therefore they are zero in the arithmetic Chow ring.

Bott-Chern forms for short exact sequences

Let $\overline{\xi}$ be a short exact sequence of log-singular Hermitian line bundles:

$$0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$$

By technical reasons it does not seem possible to define directly the Bott-Chern form for this exact sequence in the complex $\mathcal{D}_{\langle\langle D \rangle\rangle}$. Nevertheless, using the two propositions above, one can determine a Bott-Chern class, defined up to boundary in the Deligne complex, by changing the singular metrics for smooth metrics.

Arithmetic characteristic classes

Let $\phi \in B[[T_1, \dots, T_n]]$ be a symmetric power series with coefficients in a subring B of $\mathbb{R}$. Then we can attach, to every log-singular Hermitian vector bundle $\overline{E} = (E, h)$ of rank n over a pair (X, D), a characteristic class

$$\widehat{\phi}(\overline{E}) \in \widehat{\mathrm{CH}}^{*}_{B}(X, \langle\langle D \rangle\rangle).$$

Properties

1. Functoriality.
2. Compatibility with Chern forms.
3. Compatibility with change of metric.
4. Compatibility with the definition of Gillet and Soulé.

Faltings height

Let X be a projective arithmetic variety over $\mathbb{Z}$, let D be a normal crossing divisor on $X_{\mathbb{Q}}$. Let $\overline{\mathcal{L}}$ be an ample line bundle provided with a log-singular Hermitian metric. We denote by $\mathrm{Z}^p_U(X_{\mathbb{Q}})$ the group of codimension p cycles that have no component contained in D. We can define the Faltings height associated to $\overline{\mathcal{L}}$, denoted $h_{\overline{\mathcal{L}}}$ as follows.
For each $y \in \mathrm{Z}^p_U(X_{\mathbb{Q}})$ let $\overline{y}$ be its Zariski closure and let $\mathfrak{g}_y = (\omega_y, g_y)$ be any Green form for y. Then

$$h_{\overline{\mathcal{L}}} = \widehat{\deg}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^{d-p+1} \cdot (y, \mathfrak{g}_y)) - \int_{X_{\mathbb{C}}} \mathrm{c}_1(\overline{\mathcal{L}})^{d-p+1} \wedge g_y$$

A finiteness theorem

Theorem (G. Freixas)

Let X be a projective arithmetic variety over $\mathbb{Z}$, let D be a normal crossing divisor on $X_{\mathbb{Q}}$. Let $\overline{\mathcal{L}}$ be an ample line bundle provided with a log-singular Hermitian metric. Then for every constant $C \geq 0$, there exist only finitely many effective cycles $z \in \mathrm{Z}^p_U(X_{\mathbb{Q}})$ such that $\deg_{\mathcal{L}}(z) \leq C$ and $h_{\overline{\mathcal{L}}}(z) \leq C$.

Examples

Modular curves

Let $\mathfrak{H}$ denote the upper half plane with complex coordinate $z = x + iy$, and

$$X(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \cup \{S_\infty\}$$

the modular curve with the cusp S_∞.
Let $\mathcal{X}(1) = \mathbb{P}^1_{\mathbb{Z}}$ be the regular model for the modular curve $X(1)$. With s_∞ denoting the Zariski closure of (the normal crossing divisor) $S_\infty \subset X(1)$ and k a positive integer satisfying $12|k$, we define the *line bundle of modular forms of weight k* by $\mathcal{M}_k = \mathcal{O}(s_\infty)^{\otimes k/12}$. The line bundle $\mathcal{M}_k$ is equipped with the Petersson metric $\|\cdot\|$, which is a log-singular Hermitian metric along S_∞ (and the elliptic fixed points).

Modular curves

Theorem (J. B. Bost, U. Kühn)

The normalized arithmetic self intersection number is

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_k)^2) = k^2.\zeta_{\mathbb{Q}}(-1).\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

Product of modular curves

We consider the arithmetic threefold $\mathcal{H} = \mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$; we let p_1, resp. p_2 denote the projection onto the first, resp. second factor. The divisor

$$D = p_1^* \mathcal{X}(1) \times p_2^* s_\infty + p_1^* s_\infty \times p_2^* \mathcal{X}(1)$$

induces a normal crossing divisor $D_{\mathbb{R}}$ on $\mathcal{H}_{\mathbb{R}}$. For $k, l \in \mathbb{N}$, $12|k$, $12|l$, we define the Hermitian line bundle

$$\overline{\mathcal{L}}(k,l) = p_1^* \overline{\mathcal{M}}_k \otimes p_2^* \overline{\mathcal{M}}_l,$$

which is log-singular along D.
We have

$$\widehat{\deg}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}}(k,l))^3) = \frac{k^2 \cdot l + l^2 \cdot k}{4} \left(\frac{1}{2} \zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1) \right).$$

Hecke correspondence divisors

Let N be a positive integer, and M_N the set of integral (2×2)-matrices of determinant N. Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ acts from the right on the set M_N and that a complete set of representatives for this action is given by the set

$$R_N = \left\{ \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, b, d \in \mathbb{Z};\ ad = N;\ d > 0;\ 0 \leq b < d \right\}.$$

The cardinality of R_N is $\sigma(N)$.
Put

$$T_N = \left\{ (z_1, z_2) \in \mathfrak{H} \times \mathfrak{H} \,\middle|\, \exists \gamma \in M_N :\ z_1 = \gamma z_2 \right\} .$$

This defines a divisor on $X(1) \times X(1)$, whose components are graphs of Hecke correspondences.

Height of Hecke cycles

We consider the Hilbert modular form with divisor T_N:

$$s_N(z_1, z_2) = \Delta(z_1)^{\sigma(N)}\Delta(z_2)^{\sigma(N)} \prod_{\gamma \in R_N} (j(\gamma z_1) - j(z_2)) .$$

It is a section of $\mathcal{L}(12\sigma(N), 12\sigma(N))$; we put $\mathcal{T}_N = \operatorname{div}(s_N) \subseteq \mathcal{H}$.

Theorem (—, Kühn, Kramer; Autissier)

$$\operatorname{ht}_{\overline{\mathcal{L}(k,k)}}(\mathcal{T}_N) = (2k)^2 \left(\sigma(N) \left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1) \right) + \sum_{d|N} \frac{d \log(d)}{24} - \frac{\sigma(N)\log(N)}{48} \right) .$$

Hilbert modular surfaces

Let $p \equiv 1 \mod 4$ be a prime and let $\mathcal{O}_K$ be the ring of integers of $K = \mathbb{Q}(\sqrt{p})$.
Let

$$\Gamma_K(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K) \mid (a-1), b, c, (d-1) \in N.\mathcal{O}_K \right\}.$$

Write $X(N) = \mathfrak{H}^2/\Gamma_K(N)$.
Let ζ_N be a primitive root of unity. Following Deligne-Pappas $X(N)$ has a regular integral model $\mathcal{X}(N)$ defined over $\operatorname{Spec}(\mathbb{Z}[\zeta_N, 1/N])$. Moreover, following Rapoport we know that there is a regular toroidal compactification that we denote $\widetilde{\mathcal{X}}(N)$.
For k sufficiently divisible there is a line bundle $\mathcal{M}_k(\Gamma_K(N))$ on $\widetilde{\mathcal{X}}(N)$, whose global sections correspond to holomorphic Hilbert modular forms of weight k for $\Gamma_k(N)$ with Fourier coefficients in $\mathbb{Z}[\zeta_N, 1/N]$.

Hilbert modular surfaces

Let $\widetilde{X}(N)$ be the corresponding toroidal compactification of $X(N)$. We denote by $T^c_N(m)$ the Hirzebruch-Zagier divisors on $\widetilde{X}(N)$. Let $\mathcal{T}^c_N(m)$ be the Zariski closure of $T^c_N(m)$ in $\widetilde{\mathcal{X}}(N)$.
Bruinier has constructed Green functions for the divisors $T^c_N(m)$ that we denote $\mathfrak{g}_N(m)$.
We put $S_N = \operatorname{Spec} \mathbb{Z}[\zeta_N, 1/N]$ and

$$\mathbb{R}_N = \mathbb{R} \Bigg/ \left\langle \sum_{p|N} \mathbb{Q} \cdot \log(p) \right\rangle$$

Then there is a well defined degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^1(S_N) \longrightarrow \mathbb{R}_N .$$

Hilbert modular surfaces

Let Σ be the set of complex embeddings from $\mathbb{Q}(\zeta_N)$ into $\mathbb{C}$ and we let

$$D = \coprod_{\sigma\in\Sigma} (\widetilde{\mathcal{X}}(N) \setminus \mathcal{X}(N)).$$

For k sufficiently divisible we denote by $\overline{\mathcal{M}}_k(\Gamma_K(N))$ the line bundle $\mathcal{M}_k(\Gamma_K(N))$ equipped with the Petersson metric.

Proposition

There is a well defined arithmetic Chern class

$$\widehat{\mathrm{c}}_1(\overline{\mathcal{M}}_k(\Gamma_K(N))) \in \widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}(N), \langle\langle D\rangle\rangle).$$

Moreover the pairs $\widehat{\mathcal{T}}^c_N(m) = (\mathcal{T}^c_N(m), \mathfrak{g}_N(m))$ also define classes in $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}(N), \langle\langle D\rangle\rangle)$

Hilbert modular surfaces

We write symbolically

$$\widehat{c}_1(\overline{\mathcal{M}}^{\vee}_{1/2}) = -\frac{1}{2k}\widehat{c}_1(\overline{\mathcal{M}}_k(\Gamma_K(N))).$$

Theorem (Bruinier,_,Kühn)

The arithmetic generating series

$$\widehat{c}_1(\overline{\mathcal{M}}^{\vee}_{1/2}) + \sum_{m>0} \widehat{\mathcal{T}}^c_N(m) q^N$$

is a modular form of weight 2, level p and character χ_p with values in $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{X}}(N), \langle\langle D\rangle\rangle)$.

Hilbert modular surfaces

Write $d_N = [\mathbb{Q}(\zeta_N) : \mathbb{Q}] \cdot [\Gamma : \Gamma(N)]$.

Theorem (Bruinier,—,Kühn)

In $\mathbb{R}_N$ it holds the equality

$$\widehat{\deg}\, \widehat{c}_1(\overline{\mathcal{M}}(\Gamma_K(N)))^3 = - k^3 d_N \zeta_K(-1) \left(\frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log(p) \right)$$