Quaternionic Kähler manifolds

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Outline of the lecture

The holonomy group of a Riemannian manifold

Space of algebraic curvature tensors of quaternionic Kähler type and geometric consequences

Examples

The idea of parallel transport

Definition

A Riemannian manifold (M, g) is a smooth manifold M endowed with a scalar product g_x in T_xM depending smoothly on $x \in M$.

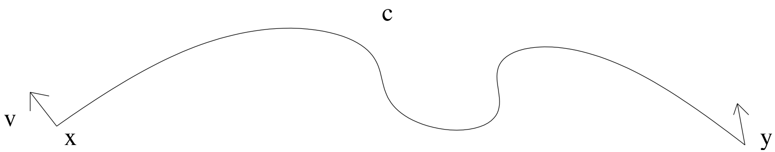
Idea of parallel transport

Associate to any curve of a Riemannian manifold M from a point x to a point y an isomorphism of the tangent spaces at x and y.

Parallel transport

Let $c : [0, 1] \to M$ be a smooth curve from x to y.

Parallel transport of vectors $v \in T_x M$ from x to y along c



defines a linear isometry

$$P_c : (T_x M, g_x) \to (T_y M, g_y).$$

The holonomy group

If $x = y$, then the curve c is a loop based at x and the parallel transport satisfies

$$P_c \in O(T_x M).$$

The subgroup

$$\mathrm{Hol}(x) := \langle P_c | c \text{ loop based at x} \rangle \subset O(T_x M) \cong O(n)$$

is called the holonomy group of (M^n, g) at x.

Independence of the base point

Let c be a curve from x to y

then the holonomy groups at x and y are related by

$$\mathrm{Hol}(x) = P_c^{-1} \mathrm{Hol}(y) P_c.$$

Hence, for connected M we do not need to specify x.

The group $\mathrm{Hol} \subset O(n)$ is well-defined up to conjugation.

Berger's list

Theorem

Let M be a complete irreducible simply connected Riemannian manifold.
Then M is a symmetric space or Hol belongs to the following list:

- $SO(n)$ *(generic case),*
- $SU(n), U(n) \subset SO(2n)$,
- $Sp(n), \quad Sp(n) \cdot Sp(1) \subset SO(4n)$,
- $G_2 \subset SO(7)$,
- $Spin(7) \subset SO(8)$.

The groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$

The groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$ act on $\mathbb{H}^n = \mathbb{R}^{4n}$.

We consider $\mathbb{H}^n$ as right vector space over the quaternions $\mathbb{H}$.

The group $Sp(n) := O(4n) \cap GL(n, \mathbb{H})$ is a compact real form of the complex symplectic group $Sp(n, \mathbb{C}) = Sp(\mathbb{C}^{2n})$.

The $Sp(1)$-factor in $Sp(n) \cdot Sp(1)$ is the group of unit quaternions acting from the right.

Classical special holonomy groups

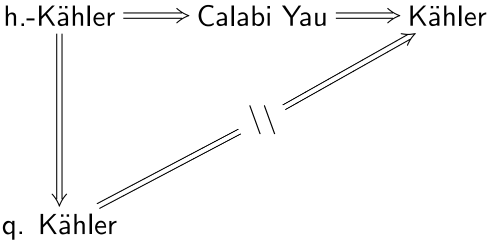
Definition

A Riemannian manifold is called

- Kähler if $\mathrm{Hol} \subset U(n)$,
- Calabi-Yau if $\mathrm{Hol} \subset SU(n)$,
- Hyper-Kähler if $\mathrm{Hol} \subset Sp(n)$,
- quaternionic Kähler if $\mathrm{Hol} \subset Sp(n) \cdot Sp(1)$ with $n > 1$.

Inclusions between classical holonomies

We have the following implications:



A (complete s.c.) non-symm. quaternionic Kähler manifold is Kähler if and only if it is already hyper-Kähler.

Geometrically the q.K. condition means that M admits a parallel subbundle $Q \subset End\ TM$ which is locally spanned by 3 anticommuting skew-symm. almost cx. structures $J_1, J_2, J_3 = J_1 J_2$. In the h.K. case the J_α are globally defined and parallel.

Algebraic curvature tensors

Situation

Given a Euclidian vector space $(V, \langle \cdot, \cdot \rangle)$ and a Lie subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$.

Definition

An algebraic curvature tensor of type $\mathfrak{g}$ is

- an element $R \in \mathfrak{g} \otimes \Lambda^2 V^*$,

$$V \times V \ni (X, Y) \mapsto R(X, Y) \in \mathfrak{g}$$

- satisfying the first Bianchi identity

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

- Denote by $\mathcal{R}(\mathfrak{g})$ the space of algebraic curvature tensors of type $\mathfrak{g}$.

Consequences of the Bianchi identity

The first Bianchi identity implies the symmetry in pairs

$$\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$$

Using the scalar product, we have $V \cong V^*$
and $\mathfrak{g} \subset so(V) \cong \Lambda^2 V \cong \Lambda^2 V^*$.
This implies $R \in S^2\mathfrak{g} \subset S^2\Lambda^2 V$.

Algebraic curvature tensors of quaternionic Kähler type

The curvature tensor of a q.K. manifold M is of type $\mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ at each $x \in M$.

Let $V = \mathbb{H}^n$ be the standard $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$-module.

Complexified it becomes a tensor-product $V^{\mathbb{C}} \cong H \otimes_{\mathbb{C}} E$,

where $H = \mathbb{C}^2$ is the standard irreducible module of $\mathfrak{sp}(1) \subset \mathfrak{sp}(\mathbb{C}^2)$

and $E = \mathbb{C}^{2n}$ is the standard irreducible module of $\mathfrak{sp}(n) \subset \mathfrak{sp}(\mathbb{C}^{2n})$.

Algebraic curvature tensors of quaternionic Kähler type II

The complex bilinear extension $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ of $\langle \cdot, \cdot \rangle$

equals $\langle \cdot, \cdot \rangle_{\mathbb{C}} = \omega_H \otimes \omega_E$,

where ω_H and ω_E are the invariant symplectic forms.

Let j_H and j_E be the invariant quaternionic structures on H and E.

V is recovered as the set of fixed-points of the antilinear involution $\rho = j_H \otimes j_E$.

Main result

Theorem

(Alekseevsky 1968, Salamon 1980)

- *It holds*

$$\mathcal{R}(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) = \mathbb{R}R_0 + \mathcal{R}(\mathfrak{sp}(n)),$$

- *where R_0 is the curvature tensor of $P^n_{\mathbb{H}}$ and*
- $\mathcal{R}(\mathfrak{sp}(n))^{\mathbb{C}} \cong S^4E.$

Sketch of the proof

Proof.

The complexification of $\mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ is
$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(H) \oplus \mathfrak{sp}(E) \underset{\omega_H, \omega_E}{\cong} S^2H \oplus S^2E$.

This implies: $S^2\mathfrak{g}^{\mathbb{C}} = S^2(S^2H \oplus S^2E) =$
$S^2S^2H + S^2S^2E + S^2H \otimes_S S^2E =$
$(\mathbb{C}B_{\mathfrak{sp}(1)} + S^2S^2E) + \underbrace{S_0^2S^2H}_{=S^4H,\text{irred.}} + \underbrace{S^2H \otimes_S S^2E}_{\text{irred.}}$.

sketch of the proof II

Proof.

The $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$-module $\mathbb{C}B_{\mathfrak{sp}(1)} + S^2S^2E$ does not contain any submodule isomorphic to $S_0^2S^2H$ or $S^2H \otimes_S S^2E$.

Therefore it suffices to prove:

1. $R_0 = aB_{\mathfrak{sp}(1)} + bB_{\mathfrak{sp}(n)} \in \mathbb{C}B_{\mathfrak{sp}(1)} + S^2S^2E$ with $a, b \in \mathbb{R}^*$,
2. $\exists$ tensor $T \in S_0^2S^2H$ s.t. $T \notin \mathcal{R}(\mathfrak{g})^{\mathbb{C}}$,
3. $\exists$ tensor $T \in S^2H \otimes_S S^2E$ s.t. $T \notin \mathcal{R}(\mathfrak{g})^{\mathbb{C}}$,
4. $S^2S^2E \cap \mathcal{R}(\mathfrak{g})^{\mathbb{C}} = S^4E$.

sketch of the proof III

Proof.

The curvature tensor of $P^n_{\mathbb{H}}$ is well-known to be

$$R_0(X,Y) = \frac{1}{2}\sum_\alpha \langle X, J_\alpha Y\rangle J_\alpha + \frac{1}{4}\left(X \wedge Y + \sum_\alpha J_\alpha X \wedge J_\alpha Y \right).$$

It is normalized s.t. $\frac{1}{4} \leq \kappa \leq 1$,

$$\kappa(X \wedge J_\alpha X) = 1 \text{ and } \mathrm{scal}_{R_0} = 4n(n+2).$$

It is easy to see

$$R_0 = n\pi_{\mathfrak{sp}(1)} + \pi_{\mathfrak{sp}(n)} : \Lambda^2 V \to \mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1).$$

The tensors $B_{\mathfrak{sp}(1)}, B_{\mathfrak{sp}(n)} \in S^2\Lambda^2 V^*, \Lambda^2 V \to \Lambda^2 V^* \underset{\langle\cdot,\cdot\rangle}{\cong} \Lambda^2 V$ are scalar multiples of $\pi_{\mathfrak{sp}(1)}$ and $\pi_{\mathfrak{sp}(n)}$.

sketch of the proof IV

Proof.

More precisely $a = -\frac{n^2}{4}, b = -\frac{1}{4(2n^2-3n+2)}$ and for $n = 1 \Rightarrow a = b = -\frac{1}{4}$. This finishes point (1).

To check (2) and (3) one can take $T = h^4$ and $h^2 e^2$ with $h \in H$ and $e \in E$.

It remains point (4). Let $(h_a)_{a=1}^{2}$ and $(e_A)_{A=1}^{2n}$ be bases of H and E. We denote by $e_{aA} = h_a \otimes e_A$ the corresponding basis of $V^{\mathbb{C}} = H \otimes E$.

We use upper indices for the dual bases.

sketch of the proof V

Proof.

$T \in S^2S^2E \cong S^2S^2E^*$ is given by

$$T = \sum T_{ABCD} e^A \otimes e^B \otimes e^C \otimes e^D,$$

where T_{ABCD} is symmetric in (A, B) and (C, D) and in the pair $((A, B,), (C, D))$.

$$\begin{aligned}
\Lambda^2(H \otimes E) &\stackrel{proj.}{\rightarrow} \omega_H \otimes S^2E \cong S^2E, \\
T(e_{aA}, e_{bB}) &= \sum_{C,D} \omega_{ab} T_{ABCD} e^C \otimes e^D \text{ and} \\
T(e_{aA}, e_{bB})e_{cC} &= \sum_{D} \omega_{ab} T_{ABCD} h_c e^D \\
&\in H \otimes E^* \underset{\omega_E}{=} H \otimes E = V^{\mathbb{C}}
\end{aligned}$$

sketch of the proof VI

Proof.

The Bianchi identity reads:

$$0 = \omega_{ab} T_{ABCD} h_c + \omega_{bc} T_{BCAD} h_a + \omega_{ca} T_{CABD} h_b$$

Choose (h_a) s.t. $\omega_{ab} := \omega_H(h_a, h_b) = \epsilon_{ab}$ and $a = 1, b = 2 = c$ in the Bianchi identity to obtain
$0 = T_{ABCD} h_2 - T_{CABD} h_2 \Leftrightarrow T_{ABCD} = T_{CABD}$.
Using the symmetries of T this implies $T \in S^4 E$.
Conversely, one can check that $T \in S^4 E$ satisfies the Bianchi identity (due to $\mathrm{dim} H = 2$.).

□

Geometric consequences

Corollary

Any q.K. manifold is Einstein, i.e. Ric = cg and c = 0 iff M is locally h.K.

Proof.

$P^n_{\mathbb{H}}$ is Einstein and

S^4E is completely trace-free with respect to $\omega_H \otimes \omega_E$, since $S^4E \cong \omega_H^2 \otimes S^4E \subset S^2\Lambda^2H \otimes S^2S^2E \subset S^2(\Lambda^2H \otimes S^2E) = S^2\Lambda^2V^{\mathbb{C}}$.

□

We will be mainly be interested in $c \neq 0$.

Geometric consequences II

Corollary

For a q.K. manifold we have

$$scal \neq 0 \Leftrightarrow \mathfrak{hol} = \mathrm{Lie\ Hol} \supset \mathfrak{sp}(1).$$

Proof.

$$R_0(J_\alpha) = n\pi_{\mathfrak{sp}(1)}J_\alpha + \pi_{\mathfrak{sp}(n)}J_\alpha = nJ_\alpha \in \mathfrak{hol}.$$

□

Geometric consequences III

Corollary

Any q.K. manifold of scal $\neq 0$ is locally irreducible.

Proof.

If $M \cong_{loc.} M_1 \times M_2$ then $\mathfrak{hol}_{loc} = \mathfrak{hol}_1 + \mathfrak{hol}_2$ and the holonomy module splits as $V = V_1 \oplus_\perp V_2$.

By the previous corollary $\mathfrak{sp}(1) \subset \mathfrak{hol} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$.

Hence $\mathfrak{hol} = \mathfrak{sp}(1) \oplus \mathfrak{h}$ and $\mathfrak{sp}(1) \subset \mathfrak{hol}_i$ for $i = 1$ or $i = 2$.

Suppose $i = 1$ then $\mathfrak{sp}(1)$ acts trivially on V_2, which is impossible, as $J_\alpha^2 = -Id$.

□

Examples: Wolf spaces

Apart from $\mathbb{H}P^n$ there is a list a compact symmetric spaces which are q.K. of positive scalar curvature,

the famous Wolf spaces.

They all can be obtained as follows:

- Let G be a cp. s.c. simple Lie group and $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$ a Cartan subalgebra,
- μ the highest root w.r.t. some system of simple roots and
- $s_\mu^{\mathbb{C}} = \mathrm{span}\{H_\mu, E_{\pm\mu}\} \subset \mathfrak{g}^{\mathbb{C}}$ the corresponding three-dimensional subalgebra.
- $H_\mu \in i\mathfrak{h}$ is normalized such that

$$[H_\mu, E_{\pm\mu}] = \pm 2E_{\pm\mu}.$$

- Then ad_{H_μ} has eigenvalues $0, \pm 1, \pm 2$.

Wolf spaces II

- This defines a grading

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

- where $\mathfrak{g}_{\pm 2} = \mathbb{C}E_{\pm\mu}$ and $\mathfrak{g}_0 = \mathbb{C}H_\mu \oplus Z_{\mathfrak{g}^{\mathbb{C}}}(s_\mu^{\mathbb{C}})$.
- We put
- $s_\mu := \mathfrak{g} \cap s_\mu^{\mathbb{C}}$,
- $\mathfrak{k} := \mathfrak{g} \cap \sum_{i=0,\pm 2} \mathfrak{g}_i = s_\mu \oplus Z_{\mathfrak{g}}(s_\mu) = N_{\mathfrak{g}}(s_\mu)$.
- $\mathfrak{m} := \mathfrak{g} \cap \sum_{i=\pm 1} \mathfrak{g}_i$.

Wolf spaces III

- Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a symmetric decomposition,
- which defines a s.c. q.K. symmetric space of cp. type

$$M = G/K,$$

- where $K = N_G(s_\mu) \subset G$ is the Lie subgroup generated by $\mathfrak{k}$.
- The holonomy of M is identified with the isotropy group

$$\mathrm{Hol} = \mathrm{Ad}_K|_{\mathfrak{m}},$$

  by the Ambrose-Singer theorem.
- The invariant quaternionic structure Q is defined by the adjoint action of $s_\mu \cong \mathrm{sp}(1)$ on $\mathfrak{m} \cong T_o M$ with $o = eK$.

Duals of Wolf spaces

- Let $M = G/K$ be a Wolf space and $\hat{G}$ the s.c. simple Lie group with
- $\mathrm{Lie}\hat{G} = \hat{\mathfrak{g}} = \mathfrak{k} + i\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$.
- Then $\hat{M} = \hat{G}/\hat{K}$ is a q.K. symm. space of non.-cp. type.
- It has negative scalar curvature.
- For $M = P^n_{\mathbb{H}}$, the quaternionic projective space, $\hat{M} = H^n_{\mathbb{H}}$ is the quaternionic hyperbolic space.

Duals of Wolf spaces II

The Wolf spaces and their duals can be characterized as follows:

Theorem (Alekseevsky-Cortés '97)

Let M be a q.K. mf. of non-zero scalar curvature admitting a transitive unimodular group of isometries. Then M is a Wolf space or dual to a Wolf space.

The duals of the Wolf space admit compact quotients which are examples of cp. q.K. mfs. of negative scalar curvature.

A complete q.K. mf. of positive scalar curvature is compact, by Myer's Thm, and there is the following:

Conjecture (Le Brun-Salamon '94)

Any complete q.K. mf. M of $\mathrm{scal} > 0$ is a Wolf space.

Status of the Lebrun-Salamon conjecture

The conjecture is proven for

- dim=4 (Hitchin '81, Friedrich-Kurke '82),
- dim=8 (Poon-Salamon '91),
- dim=12 (Herrera-Herrera '02).

Moreover there exists the following result:

Theorem Le Brun '93

For any n, there are only finitely many complete q.K. mfs. of dimension $4n$ and $\mathrm{scal} > 0$, up to isometries and rescaling.