Introduction to supersymmetry

Vicente Cortés
Institut Élie Cartan
Université Henri Poincaré - Nancy I
cortes@iecn.u-nancy.fr

August 31, 2005

Outline of the lecture

The free supersymmetric scalar field

Sigma-models

- The linear supersymmetric sigma-model
- Non-linear supersymmetric sigma-models

Extended supersymmetry, special geometry

The free supersymmetric scalar field

Let $\mathbb{M} = V = (\mathbb{R}^d, \eta =< \cdot, \cdot >)$ be a pseudo-Euclidian vector space, e.g. $\mathbb{M}$ = Minkowski space, the space-time of special relativity.

A scalar field on $\mathbb{M}$ is a function $\phi : \mathbb{M} \to \mathbb{R}$.

The simplest Lagrangian is

$$\mathcal{L}_{bos}(\phi) = \langle \text{grad}\phi, \text{grad}\phi \rangle = \eta^{-1}(d\phi, d\phi) =: |d\phi|^2.$$

It is invariant under any isometry of $\mathbb{M}$, since $d(\varphi^*\phi) = \varphi^* d\phi$, $\forall \varphi \in \text{Isom}(\mathbb{M})$.

The corresponding Euler-Lagrange equations are linear:

$$0 = \text{div grad}\phi = \Delta\phi.$$

(pseudo-Euclidian version of the Laplacian) .

The free supersymmetric scalar field II

Suppose now that we have a non-deg. bilinear form β on the spinor module S of V such that $\exists\, \sigma, \tau \in \{\pm 1\}$:

(i) $\beta(s, s') = \sigma \beta(s', s)$

(ii) $\beta(\gamma_v s, s') = \tau \beta(s, \gamma_v s')$
$\forall s, s' \in S, v \in V$, where $\gamma_v : S \to S$ is the Clifford multiplication by $v \in V$.

All such forms have been determined in [Alekseevsky-C., Comm. Math. Phys. '97].

The free supersymmetric scalar field III

If $\sigma\tau = +1$, which will be assumed from now on, we can define a symmetric vector-valued bilinear form

$$\Gamma = \Gamma_\beta : S \times S \to V$$

by the equation

$$\langle \Gamma(s, s'), v \rangle = \beta(\gamma_v s, s') \quad \forall s, s' \in S,\ v \in V.$$

Γ is equivariant with respect to the connected spin group and defines an extension of the Poincaré algebra

$$\mathfrak{g}_0 = \text{Lie Isom } (\mathbb{M}) = so(V) + V$$

to a Lie superalgebra

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \text{ with } \mathfrak{g}_1 = S.$$

The free supersymmetric scalar field IV

- It turns out that the Lagrangian $\mathcal{L}_{bos}(\phi)$ for a scalar field ϕ can be extended to a Lagrangian $\mathcal{L}(\phi, \psi)$ depending on the additional spinor field $\psi : \mathbb{M} \to S$ in such a way that the action of $Isom_0(\mathbb{M}) = SO_0(V) \ltimes V$ on scalar fields ϕ is extended to an action of its double covering $Spin_0(V) \ltimes V$ on fields (ϕ, ψ) preserving the Lagrangian $\mathcal{L}(\phi, \psi)$.
- Moreover, the infinitesimal action of $\mathfrak{g}_0$ extends, roughly speaking, to an infinitesimal action of $\mathfrak{g}$ preserving $\mathcal{L}(\phi, \psi)$ up to a divergence.

The free supersymmetric scalar field V

Formula for the Lagrangian

- $\mathcal{L}(\phi, \psi) = \eta^{-1}(d\phi, d\phi) + \beta(\psi, D\psi)$,
- where D is the Dirac operator

$$D\psi = \sum \gamma^\mu \partial_\mu \psi, \quad \gamma^\mu = \sum \eta^{\mu\nu}\gamma_\nu \text{ with } \gamma_\nu = \gamma_{\partial_\nu}.$$

- In this formula ψ has to be understood as an odd element of

$$\Gamma_A(\Sigma) := \Gamma(\Sigma) \otimes A,$$

- where $\Sigma = \mathbb{M} \times S \to \mathbb{M}$ is the trivial spinor bundle
- and $A = \Lambda E$ is the exterior algebra of some auxiliary finite dimensional vector space E.

The free supersymmetric scalar field VI

The bilinear form $\beta : S \times S \to \mathbb{R}$
extends as follows to an even $C^\infty_A(\mathbb{M})$ -bilinear form

$$\beta : \Gamma_A(\Sigma) \times \Gamma_A(\Sigma) \to C^\infty_A(\mathbb{M}) = C^\infty(\mathbb{M}) \otimes A.$$

Let (ϵ_a) be a basis of S,
$\beta_{ab} := \beta(\epsilon_a, \epsilon_b)$ and
$\psi = \sum \epsilon_a \psi_a$,
$\psi' = \sum \epsilon_a \psi'_a \in \Gamma_A(\Sigma) = \Gamma(\Sigma) \otimes A = S \otimes C^\infty_A(\mathbb{M})$.
Then on defines

$$\beta(\psi, \psi') := \sum \beta_{ab} \psi_a \psi'_b.$$

For homogeneous elements ψ, ψ' of degree $\tilde{\psi}, \tilde{\psi}' \in \{0, 1\}$ we obtain

$$\beta(\psi, \psi') = (-1)^{\tilde{\psi}\tilde{\psi}'} \sigma \beta(\psi', \psi).$$

The free supersymmetric scalar field VII

This implies

$\beta(\psi, D\psi') = \sum \beta(\psi, \gamma^\mu \partial_\mu \psi')$

$= \tau \sum \beta(\gamma^\mu \psi, \partial_\mu \psi') \equiv -\tau\beta(D\psi, \psi')$ (mod *div*)

$= -\underbrace{\tau\sigma}_{=+1}(-1)^{\widetilde{D\psi}\tilde{\psi}'(=\tilde{\psi}\tilde{\psi}')}\beta(\psi', D\psi)$

$= -(-1)^{\tilde{\psi}\tilde{\psi}'}\beta(\psi', D\psi)$.

In particular, $\beta(\psi, D\psi) \equiv -(-1)^{\tilde{\psi}}\beta(\psi, D\psi)$ (mod *div*).

Hence $\beta(\psi, D\psi)$ is a divergence if ψ is even.

The Euler-Lagrange equations are again linear:

$$\begin{cases} \Delta\phi = 0, \\ D\psi = 0. \end{cases}$$

It is easy to check the $Spin_0(V) \ltimes V$-invariance of $\mathcal{L}(\phi, \psi)$.

The free supersymmetric scalar field VIII

We are now going to check supersymmetry.

For any odd constant spinor

$$\lambda = \sum \epsilon_a \lambda^a \in S \otimes \Lambda^{odd} E \; (\cong \mathfrak{g}_1 \otimes \Lambda^{odd} E \subset (\mathfrak{g} \otimes \Lambda E)_0)$$

we define a vector field X on the the ∞-dim. vector space of fields. The value $X_{(\phi,\psi)} = (\delta\phi, \delta\psi)$ of X at (ϕ, ψ) is

$$\begin{cases} \delta\phi := -\beta(\psi, \lambda) \in C^\infty_A(\mathbb{M})_0 \\ \delta\psi := \gamma_{\mathrm{grad}_\phi} \lambda \in \Gamma_A(\Sigma)_1 \end{cases}$$

We check that this infinitesimal transformation preserves the Lagrangian up to a divergence

$$\delta\mathcal{L}(\phi, \psi) \equiv 2\eta^{-1}(d\delta\phi, d\phi) + 2\beta(\delta\psi, D\psi) \pmod{div}$$

The free supersymmetric scalar field IX

Here we have used that, by previous calculations:

$\beta(\psi, D\delta\psi) \equiv \underbrace{-(-1)^{\tilde{\psi}\widetilde{\delta\psi}}}_{=+1} \beta(\delta\psi, D\psi)$ (mod *div*) $= \beta(\delta\psi, D\psi)$.

$\eta^{-1}(d\delta\phi, d\phi) = -\eta^{-1}(\beta(d\psi, \lambda), d\phi)$

$= -\sum \eta^{\mu\nu}\beta(\partial_\mu\psi, \lambda)\partial_\nu\phi$.

$\beta(\delta\psi, D\psi) = \sum \beta(\gamma_{\mathrm{grad}_\phi}\lambda, \gamma^\mu\partial_\mu\psi)$

$= \tau \sum \beta(\gamma^\mu\gamma_{\mathrm{grad}_\phi}\lambda, \partial_\mu\psi)$

$= -\tau \sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\lambda, \partial_\mu\psi)$ (by the Clifford relation)

$+\frac{\tau}{2} \sum \beta((\gamma^\mu\gamma_{\mathrm{grad}_\phi} - \gamma_{\mathrm{grad}_\phi}\gamma^\mu)\lambda, \partial_\mu\psi)$

The free supersymmetric scalar field X

$$
\begin{aligned}
&= +\tau\sigma \sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\partial_\mu\psi,\lambda) \\
&+\tfrac{\tau}{2}\sum(\partial_\nu\phi)\beta([\gamma^\mu,\gamma^\nu]\lambda,\partial_\mu\psi) \\
&\equiv \sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\partial_\mu\psi,\lambda) \\
&-\tfrac{\tau}{2}\underbrace{\sum(\partial_\mu\partial_\nu\phi)}_{symm.}\beta(\underbrace{[\gamma^\mu,\gamma^\nu]}_{skew-symm.}\lambda,\psi) \text{ (mod } div\text{)}. \\
&= \sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\partial_\mu\psi,\lambda) \\
&= -\eta^{-1}(d\delta\phi, d\phi). \qquad \square
\end{aligned}
$$

The linear supersymmetric sigma-model

Instead of considering one scalar field ϕ and its superpartner ψ we may consider n scalar fields ϕ^i and n spinor fields ψ^i on $\mathbb{M}$.

The following Lagrangian is supersymmetric

$$\begin{aligned} & \mathcal{L}(\phi^1, \dots, \phi^n, \psi^1, \dots, \psi^n) \\ = & \sum_{i,j=1}^{n} g_{ij}(\eta^{-1}(d\phi^i, d\phi^j) + \beta(\psi^i, D\psi^j)) \end{aligned}$$

where g_{ij} is a constant symmetric matrix, which we assume to be non-degenerate.

The above Lagrangian is called linear supersymm. sigma-model.

The Euler Lagrange equations for the scalar fields imply that the map

$$\phi = (\phi^1, \dots, \phi^n) : \mathbb{M} \to \mathbb{R}^n$$

is harmonic, where the target carries the metric $g = (g_{ij})$.

Bosonic supersymmetric non-linear sigma-models

It is natural to consider maps

$$\phi : \mathbb{M} \to (M, g)$$

into a curved pseudo-Riemannian manifold (M, g) and to ask:

Does there exist a supersymmetric extension $\mathcal{L}(\phi, \psi)$ of the so-called non-linear bosonic sigma-model $\mathcal{L}_{bos}(\phi) = |d\phi|^2 := (g_\phi \otimes \eta^{-1})(d\phi, d\phi)$?

The Euler-Lagrange equation of $\mathcal{L}_{bos}$ is the harmonic map equation for ϕ.

Such a model is called a supersymm. non-linear sigma-model.

One cannot expect this to exist for arbitrary target (M, g).

Restrictions on the target geometry

Supersymmetry imposes restrictions on the target geometry, which depend on the dimension d of space-time and on the signature of the space-time metric η.

In the case of 4-dimensional Minkowski space the restriction is that (M, g) is (pseudo-)Kähler.

The corresponding supersymmetric sigma-model is of the form

$$\mathcal{L}(\phi, \psi) = (g_\phi \otimes \eta^{-1})(d\phi, d\phi) + (g_\phi \otimes \beta)(\psi, D^\phi \psi) + Q(\phi, \psi),$$

where $\psi \in \Gamma_A(\phi^* TM \otimes_{\mathbb{R}} \Sigma)$, $\psi = \psi^{1,0} \oplus \overline{\psi^{1,0}}$,

with $\psi^{1,0} \in \Gamma_A(\phi^* TM^{1,0} \otimes_{\mathbb{C}} \Sigma)$ and $D^\phi = \sum \gamma^\mu \nabla^\phi_{\partial_\mu}$,

where ∇^ϕ is the natural connection in $\phi^* TM \otimes \Sigma$ and

Q is a term quartic in the fermions constructed out of the curvature-tensor R^g of g using that (M, g) is Kähler and $S = \mathbb{C}^2$.

Extended supersymmetry, special geometry

- The super-Poincaré algebra $\mathfrak{g} = \mathfrak{g}_{N=1} = \mathfrak{g}_0 + \mathfrak{g}_1$ underlying the above supersymmetric NLSM on 4-dim. Minkowski space

  is minimal in the sense that $\mathfrak{g}_1 = S$ is an irreducible $Spin(1,3)$-module.
- There exists another super-Poincaré algebra $\mathfrak{g} = \mathfrak{g}_{N=2} = \mathfrak{g}_0 + \mathfrak{g}_1$ for which $\mathfrak{g}_1 = S \otimes \mathbb{R}^2$ is the sum of two irreducible submodules.

  Remark: $\mathfrak{g}_{N=1}$ is not a subalgebra of $\mathfrak{g}_{N=2}$.

  In fact, the $Spin(V)$-submodules $S \otimes v$, $v \in \mathbb{R}^2$, are commutative subalgebras, i.e. $[S \otimes v, S \otimes v] = 0$.

Field theories admitting the extended super-Poincaré algebra $\mathfrak{g}_{N=2}$ as supersymmetry algebra are called

$N = 2$ supersymmetric theories.

The target geometry of such theories is called special geometry.

The geometry depends on the field content of the theory.

There are two fundamental cases:

(a) Theories with vector multiplets $\Longrightarrow$ target geometry is (affine) special Kähler.

(b) Theories with vector hyper multiplets $\Longrightarrow$ target geometry is hyper-Kähler.