The special geometry of Euclidian supersymmetry

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Outline of the lecture

Special para-Kähler manifolds

Maps between special geometries from dimensional reduction

- Dimensional reduction from 5 to 4 dimensions
- Dimensional reduction from 4 to 3 dimensions

Motivation

- This talk is based on joint work with C. Mayer, T. Mohaupt and F. Saueressig (ITP, University of Jena):
  - Special Geometry of Euclidean Supersymmetry I: Vector Multiplets, J. High Energy Phys. 03 (2004) 028, hep-th/0312001.
  - Special geometry of Euclidean supersymmetry II: hypermultiplets and the c-map, J. High Energy Phys. 06 (2005) 025, hep-th/0503094
- For the Euclidian 4-space there exists an $N = 2$ super-Poincaré algebra.
- There exists no $N = 1$ algebra on the Euclidian 4-space.
- Euclidian vector multiplets can be defined.
- The corresponding special geometry is (affine) special para-Kähler geometry.

Special para-Kähler manifolds

Definition

A para-Kähler manifold is a pseudo-Riemannian manifold (M, g) endowed with a parallel skew-symmetric involution $J \in \Gamma(End TM)$.

A special para-Kähler manifold (M, J, g, ∇) is a para-Kähler manifold (M, J, g) endowed with a flat torsion-free connection ∇ satisfying

(i) $\nabla\omega = 0$, where $\omega = g(J\cdot, \cdot)$ is the symplectic form associated to (M, J, g) and

(ii) $(\nabla_X J)Y = (\nabla_Y J)X, \quad \forall X, Y \in \Gamma(TM)$.

From the definition of a para-Kähler manifold it follows that the eigen-distributions $T^{\pm}M$ of J are isotropic, of the same dimension and integrable.

In particular, $\dim M = 2n$ and g is of split signature (n, n).

Definition

A field of involutions on a manifold M with integrable eigen-distributions of same dimension is called a para-complex structure.

A manifold endowed with a para-complex structure is called para-complex manifold.

A map $\phi : (M, J) \to (M', J')$ between para.-cx. mfs is called para-holomorphic if $d\phi \circ J = J' \circ d\phi$.

A para-holomorphic function is a para-holomorphic map $f : (M, J) \to C$ with values in the ring of para-complex numbers $C = \mathbb{R}[e], e^2 = 1$.

For any $p \in M$ there exists an open neighbourhood U and para-holomorphic functions

$$z^i : U \to C, \quad i = 1, \ldots, n = \frac{\dim M}{2},$$

such that the map $(z^1, \ldots, z^n) : U \to C^n = \mathbb{R}^{2n}$ is a diffeomorphism on its image.

Such a system of para-holomorphic functions is called a system of para-holomorphic coordinates.

Extrinsic construction of special para-Kähler manifolds

Consider the free module $V = C^{2n}$ with its global linear para-holomorphic coordinates (z^i, w_i),

its standard para-hol. symplectic form

$$\Omega = \sum dz^i \wedge dw_i$$

and the standard anti-linear involution $\tau : V \to V$ with the set of fixed points $V^\tau = \mathbb{R}^{2n}$.

We define a constant para-Kähler metric by

$$g_V(X, Y) := \mathrm{Re}(e\Omega(X, \tau Y)), \quad X, Y \in V.$$

Definition

Let (M, J) be a para-complex manifold of real dimension $2n$.

A para-hol. immersion $\phi : M \to V = C^{2n}$ is called para-Kählerian (resp. Lagrangian) if $\phi^* g_V$ is non-degenerate (resp. if $\phi^*\Omega = 0$).

It is easy to see that the metric $g = \phi^* g_V$ induced by a para-Kählerian immersion is para-Kählerian.

Lemma

Let $\phi : M \to V$ be a para-Kählerian Lagrangian immersion and $\omega = g(J\cdot, \cdot)$ the corresponding symplectic structure. Then

$\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$, *where*

$\tilde{x}^i = x^i \circ \phi$, $\tilde{y}_i = y_i \circ \phi$, $x^i = Re\, z^i$, $y^i = Re\, w^i$.

By the lemma $(\tilde{x}^i, \tilde{y}_i)$ defines a system of loc. coordinates.

Therefore, there exists a unique flat and torsion-free connection ∇ on M for which $\tilde{x}^i$ and $\tilde{y}_i$ are affine.

Theorem

Let $\phi : M \to V$ be a PKLI with induced data (J, g, ∇).

- *Then (M, J, g, ∇) is a special para-Kähler manifold.*
- *Conversely, any s.c. special para-Kähler manifold (M, J, g, ∇) admits a PKLI with induced data (J, g, ∇).*
- *Moreover, the PLKI ϕ is unique up to an element of $Aff_{Sp(\mathbb{R}^{2n})}(V)$.*

Proof of "$\Rightarrow$".

Let $\phi : M \to V$ be a PKLI with ind. data (J, g, ∇).

We have to show that (M, J, g, ∇) is special para-Kähler.

We know that (M, J, g) is para-Kähler and that ∇ is flat and torsion-free.

Proof (continued)

By the lemma, the symplectic form ω has constants coefficients w.r.t. ∇-affine coordinates $(\tilde{x}^i, \tilde{y}_i)$.

Thus $\nabla\omega = 0$. It remains to show that ∇J is symmetric.

For a ∇-parallel one-form ξ we calculate:

$$\begin{aligned} d(\xi \circ J)(X, Y) &\underset{T^\nabla=0}{=} \nabla_X(\xi \circ J)Y - \nabla_Y(\xi \circ J)X \\ &= \xi(\nabla_X(J)Y - \nabla_Y(J)X). \end{aligned}$$

Therefore, it is sufficient to prove $\xi \circ J$ is closed for $\xi = d\tilde{x}^i$ and $\xi = d\tilde{y}_i$. We check this, for example, for $\xi = d\tilde{x}^i$.

Proof (continued).

$\tilde{x}^i$ is the real-part of the para-hol. function $\tilde{z}^i = z^i \circ \phi$.

So $d\tilde{z}^i = d\tilde{x}^i + e d\tilde{x}^i \circ J$. Since $d\tilde{x}^i$ and $d\tilde{z}^i$ are closed, this shows that $d\tilde{x}^i \circ J$ is closed.

□

Corollary

Let $F : U \to C$ be a para-hol. function defined on a open set $U \subset C^n$ satisfying the non-degeneracy condition $det\ Im \frac{\partial^2}{\partial z^i \partial z^j} F \neq 0$.

- *Then $\phi_F = dF\ :\ U \to C^{2n}$*

$$z = (z^1, \ldots, z^n) \mapsto (z, \frac{\partial F}{\partial z^1}(z), \ldots, \frac{\partial F}{\partial z^n}(z))$$

  is a PKLI and hence defines a special para-K. manifold M_F.
- *Conversely, any special para-K. manifold is locally of this form.*

Dimensional reduction

Dimensional reduction is a procedure for the construction of a field theory in d space-time dimensions from one in $d+1$ dimensions.

Natural questions

- Is it possible to construct $N=2$ supersymmetric field theories with vector multiplets on 4-dimensional Euclidian space from field theories on 5-dimensional Minkowski space?
- Is it possible to construct Euclidian supersymmetric field theories in 3 dimensions out of $N=2$ supersymmetric field theories with vector multiplets in 4 dimension?

Dimensional reduction from 5 to 4 dimensions

- The allowed target geometry for the scalar fields in the relevant supersymmetric theories on 5-dimensional Minkowski space is very special (real).
- It is defined by a real cubic polynomial $h(x^1, \ldots, x^n)$ with non-degenerate Hessian $\partial^2 h$ on some domain $U \subset \mathbb{R}^n$.
- We found that dimensional reduction of such a Minkowskian theory over time yields a Euclidian $N = 2$ supersymm. theory with VMs such that the target is special para-Kähler.
- This means we get a map

$$\{\text{very special real mfs.}\} \xrightarrow{r_{4+0}^{4+1}} \{\text{special para-Kähler mfs.}\}$$

- which we call the para-r-map.

Theorem

- *There exists a map* r^{4+1}_{4+0} *which associates a special para-Kähler structure on the domain* $\widetilde{U} = U + e\mathbb{R}^n \subset C^n$ *to any very special manifold* $(U, \partial^2 h)$, $U \subset \mathbb{R}^n$.
- *The special para-Kähler structure is defined by the para-hol. fct.*

$$F : \widetilde{U} \to C, \quad F(z^1, \dots z^n) := \frac{1}{2e} h(z^1, \dots, z^n),$$

which satisfies $\det Im\, \partial^2 F \neq 0$.

This is the para-version of the r-map:

$$\{\text{very special real mfs.}\} \xrightarrow{r^{4+1}_{3+1}} \{\text{special pseudo-Kähler mfs.}\}$$

defined by B. de Wit and A. van Proeyen in 1992.

Dimensional reduction from 4 to 3 dimensions

We found two ways of constructing Euclidian supersymmetric field theories in 3 dimensions out of $N = 2$ theories with vector multiplets in 4 dimensions.

One can start with a Minkowskian theory and reduce over time or with a Euclidian theory.

This gives us two maps

$$\{\text{special pseudo.-K. mfs.}\} \xrightarrow{c^{3+1}_{3+0}} \{\text{special para-hyper-K. mfs.}\},$$

$$\{\text{special para-K. mfs.}\} \xrightarrow{c^{4+0}_{3+0}} \{\text{special para-hyper-K. mfs.}\},$$

which we call the para-c-maps. They are para-variants of the c-map, worked out by Cecotti, Ferrara and Girardello in 1989.

Para-hyper-Kähler manifolds

Definition

A para-hyper-Kähler manifold is a pseudo-Riemannian manifold with three pairwise anticommuting parallel skew-symmetric endomorphisms

- $J_1, J_2, J_3 = J_1 J_2 \in \Gamma(End\ TM)$ such that
- $J_1^2 = J_2^2 = -J_3^2 = Id$.

- A pseudo-Riem. manifold is para–hyper-Kähler iff

$$\begin{aligned} \mathrm{Hol} \quad &\subset \quad Sp(\mathbb{R}^{2n}) = Id_{\mathbb{R}^2} \otimes Sp(\mathbb{R}^{2n}) \\ &\subset \quad SO(\mathbb{R}^2 \otimes \mathbb{R}^{2n}, \omega_{\mathbb{R}^2} \otimes \omega_{\mathbb{R}^{2n}}) = SO(2n, 2n). \end{aligned}$$

- In particular, the dimension is divisible by 4.

The para-c-maps

Now I describe the para-h.-K. mf. associated to a special para-Kähler mf. (M, J, g, ∇) via the para-c-map c^{4+0}_{3+0}.

Let $N = T^*M$ be the total space of the ctg. bdl. $\pi : N \to M$ and consider the decomposition $T_\xi N = \mathcal{H}^\nabla_\xi \oplus T^v_\xi N$, $\xi \in N$, into horizontal and vertical subbundles with respect to the connection ∇.

This defines a canonical identification

$$T_\xi N \cong T_x M \oplus T^*_x M, \quad x = \pi(\xi).$$

The para-c-maps (continued)

With respect to the above identification we define a pseudo-Riemannian metric g_N on N by

$$g_N := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

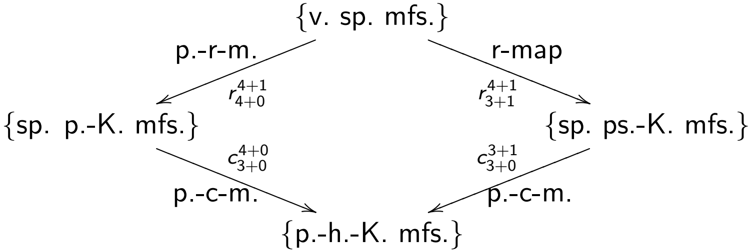
and two involutions J_1, J_2 by

$$J_1 := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \text{ and } J_2 := \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Theorem

For any special para-Kähler manifold (M, J, g, ∇), $(N, g_N, J_1, J_2, J_3 = J_1 J_2)$ is a para-hyper-Kähler manifold.

The maps between special geometries induced by dimensional reduction are summarized in the following diagram:



The diagramm is essentially commutative:

Theorem

For any very special manifold $L = (U, \partial^2 h)$ *the para-h.K. mfs.* $c^{4+0}_{3+0} \circ r^{4+1}_{4+0}(L)$ *and* $c^{3+1}_{3+0} \circ r^{4+1}_{3+1}(L)$ *are canonically isometric.*