Hyperbolic dimension

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Outline

Topological dimension

Definition 1

The *covering dimension* $\dim_{cov} X$ of a metric space X is the minimal integer n such that for every $\varepsilon > 0$ there is an open covering $\mathcal{U}$ of X with multiplicity $\leq n+1$ and $\sup_{U \in \mathcal{U}} \operatorname{diam} U \leq \varepsilon$.

Definition 2

The *coloured dimension* $\dim_{col} X$ of a metric space X is the minimal integer n such that for every $\varepsilon > 0$ there is a covering $\mathcal{U}$ of X consisting of $n+1$ open subsets U_j, $j = 0, \ldots, n$, such that:

- $U_j = \bigcup_{\alpha \in I_j} U_{j\alpha} \quad \forall j$;
- $U_{j\alpha} \cap U_{j\alpha'} = \emptyset \quad \forall \alpha \neq \alpha'$;
- $\operatorname{diam} U_{j\alpha} \leq \varepsilon \quad \forall j, \alpha$.

Definition 3

The *polyhedral dimension* $\dim_{pol} X$ via simplicial complexes.

Topological dimension

Proposition

Let X be a metric space. Then

$$\dim_{cov} X = \dim_{col} X = \dim_{pol} X.$$

The common value is called *topological dimension*, $\dim X$.

Idea of the proof

- It is clear that $\dim_{cov} X \leq \dim_{col} X$.
- Then, $\dim_{col} X \leq \dim_{pol} X$ is proven with the help of the barycentric subdivision.
- Finally, a simplicial complex, the nerve of a covering, can be constructed, which leads to $\dim_{pol} X \leq \dim_{cov} X$.

Asymptotic dimension [Gromov, 1993]

Definition

The *asymptotic dimension* $\operatorname{asdim} X$ of a metric space X is the minimal integer n such that for every $d > 0$ there is a covering $\mathcal{U}$ of X consisting of $n+1$ subsets U_j, $j = 0, \ldots, n$, such that

- $U_j = \bigcup_{\alpha \in I_j} U_{j\alpha} \quad \forall j$;
- $\exists\, D \geq 0$ such that $\operatorname{diam} U_{j\alpha} \leq D \quad \forall j, \alpha$ (D-*bounded* or *uniformly bounded*);
- $\operatorname{dist}(U_{j\alpha}, U_{j\alpha'}) \geq d \quad \forall \alpha \neq \alpha'$ (d-*disjoint*).

Asymptotic dimension [Gromov, 1993]

Proposition

Let X be a metric space. Then the following are equivalent:

- $\operatorname{asdim} X = n$.
- There is a minimal integer n such that for every $d > 0$ there exists a uniformly bounded covering of X so that no ball of radius d in X meets more than $n + 1$ elements of the cover (d-*multiplicity*).

Furthermore there are:

- A similar statement using multiplicity and Lebesgue number.
- A characterisation via simplicial complexes.

Hyperbolic dimension [Buyalo/Schroeder, 2004]

Definition

A metric space X is called *large-scale doubling* if there exist $N \in \mathbb{N}$ and $R \in \mathbb{R}^+$ such that every ball of radius $r \geq R$ in X can be covered by N balls of radius $\frac{r}{2}$.

Results

- The property to be large-scale doubling can be iterated.
- It is a quasi-isometry invariant.
- A space that is large-scale doubling has polynomial growth rate.

Hyperbolic dimension [Buyalo/Schroeder, 2004]

Definition

The *hyperbolic dimension* of a metric space X, $\operatorname{hypdim} X$, is the minimal integer n such that for every $d > 0$ there are an $N \in \mathbb{N}$ and a covering of X so that:

- no ball of radius d in X meets more than $n+1$ elements of the cover;
- there is $R \in \mathbb{R}^+$ such that any set of the covering is large-scale doubling with parameters N and R;
- any finite union of elements of the covering is large-scale doubling with parameter N.

Remark

As before, there are equivalent formulations based on multiplicity and Lebesgue number, d-multiplicity, and simplicial complexes, respectively.

Hyperbolic dimension [Buyalo/Schroeder, 2004]

Observations

- If a metric space X is large-scale doubling, then $\operatorname{hypdim} X = 0$.
- A metric space X is large-scale doubling with parameters $N = 1$ and $R \iff \operatorname{diam} X = \frac{R}{2}$.
- We get asdim if we ask for the fixed value $N = 1$ in the definition of hypdim.
- Therefore we have $\operatorname{hypdim} X \leq \operatorname{asdim} X$ for any metric space X.

Hyperbolic dimension [Buyalo/Schroeder, 2004]

Further results

- The hyperbolic dimension is a quasi-isometry invariant.
- Monotonicity: If $f : X \to X'$ is a quasi-isometric map between metric spaces X, X', then
$$\operatorname{hypdim} X \leq \operatorname{hypdim} X'.$$
- Product theorem: For any metric spaces X_1 and X_2, one has
$$\operatorname{hypdim}(X_1 \times X_2) \leq \operatorname{hypdim} X_1 + \operatorname{hypdim} X_2.$$
- For the n-dimensional hyperbolic space $\mathbb{H}^n$ one has $\operatorname{hypdim} \mathbb{H}^n = n$.
- And finally, one can show that $\mathbb{H}^n$ cannot be embedded quasi-isometrically into a $(n-1)$-fold product of trees and some euclidean factor $\mathbb{R}^N$.

Large-scale structures [Dydak/Hoffland, 2006]

Preliminary definitions

- $\mathrm{St}(A, \mathcal{B}) := \bigcup_{B \in \mathcal{B}, B \cap A \neq \emptyset} B \quad \in \mathcal{P}(X)$;
- $\mathrm{St}(\mathcal{A}, \mathcal{B}) := \{\mathrm{St}(A, \mathcal{B}) \mid A \in \mathcal{A}\} \quad \in \mathcal{P}(\mathcal{P}(X))$;
- $\mathrm{e}(\mathcal{B}) := \mathcal{B} \cup \{\{x\} \mid x \in X\} \quad \in \mathcal{P}(\mathcal{P}(X))$;
- Let $\mathcal{A}$, $\mathcal{B} \in \mathcal{P}(\mathcal{P}(X))$ such that $\forall\, B \in \mathcal{B}\ \exists\, A \in \mathcal{A}$ with $B \subset A$. Then $\mathcal{B}$ is called *refinement* of $\mathcal{A}$.

Definition

An element $\mathfrak{A} \in \mathcal{P}^3(X)$ is a *large-scale structure* on X if the following conditions hold:

- $\mathcal{B} \in \mathfrak{A}$, $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{A}$ refinement of $\mathrm{e}(\mathcal{B})$ $\implies \mathcal{A} \in \mathfrak{A}$;
- $\mathcal{A}$, $\mathcal{B} \in \mathfrak{A} \implies \mathrm{St}(\mathcal{A}, \mathcal{B}) \in \mathfrak{A}$.

Large-scale structures [Dydak/Hoffland, 2006]

Example

A large-scale structure $\mathfrak{A}$ for a metric space X is given by:

$$\mathcal{B} \in \mathfrak{A} \iff \exists M > 0 \text{ such that } \operatorname{diam} B \leq M \quad \forall B \in \mathcal{B}.$$

Definition

Let X be a space and $\mathfrak{A}$ a large-scale structure on X. The *large-scale dimension* $\dim(X, \mathfrak{A})$ is the minimal n so that $\mathfrak{A}$ is generated by a set of families $\mathcal{B}$ such that the multiplicity of each $\mathcal{B}$ is at most $n + 1$.

Thereby we say that $\mathfrak{A}$ is generated by a set of families $\mathcal{B}$ if $\mathfrak{A}$ contains all refinements of trivial extensions of all families $\mathcal{B}$.