Non-technical survey of derived FM transforms in algebraic geometry

1. Derived equivalences of projective varieties and classification

2. From automorphisms to autoequivalences

3. Birational geometry from the derived point of view

$\mathcal{A}$ = **abelian category**

$\rightsquigarrow$ can speak of short exact sequences:

$$0 \to A \to B \to C \to 0$$

$\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ = **bounded derived category**

– objects: bounded complexes $A^\bullet$, $A^i \in \mathcal{A}$

$$\ldots \to 0 \to A^{i-1} \to A^i \to \ldots \to A^j \to A^{j+1} \to 0 \ldots$$

– morphisms: $A^\bullet \to B^\bullet$ in $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ given by

$$A^\bullet \xleftarrow{\text{qis}} C^\bullet \longrightarrow B^\bullet$$

The abelian category is a full subcategory of its bounded derived category:

$$\mathcal{A} \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{A}), A \mapsto (\ldots \to 0 \to A^0 = A \to 0 \to \ldots)$$

$\mathrm{D^b}(\mathcal{A})$ is a **triangulated category**:

– <u>shift functor</u>: $\mathrm{D^b}(\mathcal{A}) \to \mathrm{D^b}(\mathcal{A})$, $A^\bullet \mapsto A^\bullet[1]$.

$$A^\bullet : \quad \cdots \longrightarrow A^{i-1} \xrightarrow{d} A^i \xrightarrow{d} A^{i+1} \longrightarrow \cdots$$

$$A^\bullet[1] : \quad \cdots \longrightarrow A^{i} \xrightarrow{-d} A^{i+1} \xrightarrow{-d} A^{i+2} \longrightarrow \cdots$$

– <u>distinguished triangles</u>:

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$$

Example: Cone construction

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$$

$$C(f)^i = A^{i+1} \oplus B^i$$

$$d : a \oplus b \mapsto -d(a) \oplus (f(a) + d_B(b))$$

X= **smooth projective variety**

$\mathcal{A} = \mathbf{Coh}(X) =$ category of coherent sheaves

Examples:

- algebraic/holomorphic vector bundles

- $Y \subset X \rightsquigarrow I_Y$, $\mathcal{O}_Y$ ideal and structure sheaf

- $x \in X \rightsquigarrow k(x) = \mathcal{O}_{\{x\}}$ skyscraper sheaf

Gabriel (1962):

$$\boxed{\mathbf{Coh}(X) \cong \mathbf{Coh}(Y) \Leftrightarrow X \cong Y.}$$

$X=$ **smooth projective curve**

$$\mathbf{D^b}(X) \;\cong\; \bigoplus \mathbf{Coh}(X)[i]$$

$$\mathcal{F}^\bullet \;\cong\; \bigoplus \mathcal{H}^j(\mathcal{F}^\bullet)[-j].$$

$$\mathbf{D^b}(X) \cong \mathbf{D^b}(Y) \;\stackrel{*}{\Rightarrow}\; \mathbf{Coh}(X) \cong \mathbf{Coh}(Y)$$

$$\Rightarrow\; X \cong Y$$

Example $E =$ elliptic curve $= \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{C}$.

$\mathcal{P} := \mathcal{O}(\boldsymbol{\Delta} - e \times E - E \times e) =$ Poincaré line bundle on $E \times E$.

$\Phi_{\mathcal{P}} : \mathbf{D^b}(E) \xrightarrow{\sim} \mathbf{D^b}(E)$, $\mathcal{F} \mapsto p_*(q^*\mathcal{F}^\bullet \otimes \mathcal{P})$.

General structure

Serre functor Equivalence

$$S_X : \mathbf{D}^{\mathbf{b}}(X) \to \mathbf{D}^{\mathbf{b}}(X), \quad \mathcal{F}^\bullet \mapsto \mathcal{F}^\bullet \otimes \omega_X[n],$$

such that

$$\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong \mathrm{Hom}_{\mathbf{D}}(\mathcal{G}^\bullet, S_X(\mathcal{F}^\bullet))^*$$

Fact Serre functors are preserved under equivalences.

Suppose $\mathbf{D}^{\mathbf{b}}(X) \cong \mathbf{D}^{\mathbf{b}}(Y)$.

Corollary 1 $\dim(X) = \dim(Y)$.

Corollary 2 $\omega_X^{\otimes k} \cong \mathcal{O}_X \Leftrightarrow \omega_Y^{\otimes k} \cong \mathcal{O}_Y$

Corollary 3 $HH^*(X) \cong HH^*(Y)$, $HH_*(X) \cong HH_*(Y)$.

Corollary 4 $R(X) \cong R(Y)$.

Classification

• $\mathbf{Kod}(X) = -\infty$, e.g. $\mathbb{P}^n$, $\mathbb{P}^n \times X$

• $\mathbf{Kod}(X) = 0$, e.g. K3, abelian varieties

⋮

• $\mathbf{Kod}(X) = \dim(X)$, 'general type'.

• $\omega_X < 0$: Fano varieties ($\mathbb{P}^n$, $\mathbb{P}^k \times \mathbb{P}^l$,...)

• $\omega_X \sim 0$: AV, CY, HK.

• $\omega_X > 0$: ball quotients, hypersurfaces in $\mathbb{P}^n$ of degree $d > n+1$,...

Bondal, Orlov (2001) Suppose $\omega_X < 0$ or > 0. If $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y) \Rightarrow X \cong Y$.

The interesting ones

X= compact Kähler/projective manifold with $c_1(X)_{\mathbb{R}} = 0$

Beauville, Bogomolov, Yau,.... There exists a finite étale cover $\widetilde{X} \to X$ such that

$$\widetilde{X} \cong (\mathbb{C}^n/\Gamma) \times \prod X_i \times \prod Y_i.$$

with unique $X_i =$ CY and $Y_i =$ HK.

Open questions

– $$\mathbf{D}^{\mathrm{b}}(X) \cong \mathbf{D}^{\mathrm{b}}(X') \overset{?}{\Rightarrow} \mathbf{D}^{\mathrm{b}}(\prod X_i) \cong \mathbf{D}^{\mathrm{b}}(\prod X'_i)$$

$$\overset{?}{\Rightarrow} \mathbf{D}^{\mathrm{b}}(X_i) \cong \mathbf{D}^{\mathrm{b}}(X'_{\sigma(i)})$$

– $X, X' =$ CY (HK): $\mathbf{D}^{\mathrm{b}}(X) \cong \mathbf{D}^{\mathrm{b}}(X') \Leftrightarrow$????.

Mukai, Orlov, Polishchuk (1981-2002) Complete theory for abelian varieties.

E.g. $\mathbf{D}^{\mathrm{b}}(A) \cong \mathbf{D}^{\mathrm{b}}(B) \Rightarrow A \times \widehat{A} \cong B \times \widehat{B}$.

Fourier–Mukai transforms

$f : X \to Y \rightsquigarrow f^* : \mathbf{Coh}(Y) \to \mathbf{Coh}(X)$ right exact and $f_* : \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$ left exact.

Derived functors $Lf^* : \mathbf{D^b}(Y) \to \mathbf{D^b}(X)$ and $Rf_* : \mathbf{D^b}(X) \to \mathbf{D^b}(Y)$.

FM transform $X \xleftarrow{q} X \times Y \xrightarrow{p} Y$, $\mathcal{P} \in \mathbf{D^b}(X \times Y) \rightsquigarrow$

$$\Phi_{\mathcal{P}} : \mathbf{D^b}(X) \longrightarrow \mathbf{D^b}(Y)$$

$$\mathcal{E}^\bullet \longmapsto Rp_*(Lq^* \otimes \mathcal{P})$$

Examples

- $f \rightsquigarrow Rf_* = \Phi_{\mathcal{O}_{\Gamma_f}}$ with $\Gamma_f =$ graph of f. E.g. for $f = \mathrm{id}$ one has $\Gamma_f = \mathcal{O}_\Delta$ with $\Delta \subset X \times X$ diagonal.

- $L =$ line bundle on $X \rightsquigarrow L \otimes (\quad) \cong \Phi_{L_{\Gamma_f}}$.

Orlov(1997) Every equivalence $F : \mathbf{D^b}(X) \cong \mathbf{D^b}(Y)$ is isomorphic to some $\Phi_{\mathcal{P}}$. The FM-kernel $\mathcal{P}$ is uniquely determined.

Autoequivalences

$\mathbf{Aut}(X) := \{f : X \cong X\} \rightsquigarrow$

$\mathbf{Aut}(\mathbf{D^b}(X)) := \{\Phi_{\mathcal{P}} : \mathbf{D^b}(X) \cong \mathbf{D^b}(X)\}$

Examples

- $[1] \cong \Phi_{\mathcal{O}_\Delta[1]}$.
- $f : X \cong X \rightsquigarrow f_* \cong \Phi_{\mathcal{O}_{\Gamma_f}}$.
- $L =$ line bundle $\rightsquigarrow L \otimes (\quad) \cong \Phi_{L_{\Gamma_f}}$.

Bondal, Orlov (2001) Suppose $\omega_X < 0$ or $\omega > 0$. Then there are no others.

Mukai, Orlov, Polishchuk For AV the situation is slightly more interesting and fully understood.

Question What happens for CY and HK???

Spherical objects and spherical twists

$X =$ CY $\Rightarrow H^*(X, \mathcal{O}) \cong H^*(S^n, \mathbb{C})$.

$\mathcal{E}^\bullet$ is **spherical** $\Leftrightarrow H^*(X, \mathcal{E}nd(\mathcal{E}^\bullet)) \cong H^*(S^n, \mathbb{C})$.

$\mathcal{E}^\bullet \rightsquigarrow \mathcal{P}_{\mathcal{E}^\bullet} := C(\mathcal{E}^{\bullet *} \boxtimes \mathcal{E}^\bullet \stackrel{\text{tr}}{\to} \mathcal{O}_\Delta) \in \mathbf{D}^{\mathbf{b}}(X \times X)$

Seidel, Thomas (2001) If $\mathcal{E}^\bullet$ is spherical, then $T_{\mathcal{E}^\bullet} := \Phi_{\mathcal{P}_{\mathcal{E}^\bullet}} : \mathbf{D}^{\mathbf{b}}(X) \cong \mathbf{D}^{\mathbf{b}}(X)$.

There exists also a relative notion of EZ-spherical twists (see Horja 2005).

Examples

- Any line bundle.

- $\mathcal{O}_C$ with C a smooth $(-1,-1)$-curve in CY threefold.

- $\mathcal{O}_C$ with C a smooth (-2)-curve in a K3.

$\mathbf{A_m}$-configuration of spherical objects

Spherical objects $\mathcal{E}_1, \ldots, \mathcal{E}_m \in \mathbf{D^b}(X)$ with

$$\dim H^*(X, \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_j)) = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Seidel, Thomas (2001)

Then $T_i := T_{\mathcal{E}_i}$ satisfy

$$T_i \circ T_j \cong T_j \circ T_i \quad \text{if} \quad |i-j| > 1$$

and

$$T_i \circ T_{i+1} \circ T_i \cong T_{i+1} \circ T_i \circ T_{i+1}.$$

Thus

$$\rho : B_m \to \mathbf{Aut}(\mathbf{D^b}(X)).$$

If $\dim(X) \geq 2$, then ρ is injective.

Example Resolution of A_m-singularity

$$x^2 + y^2 + z^{m+1} = 0$$

produces rational curves $C_1, .., C_m$. The $\mathcal{O}_{C_i}$ are spherical and $\mathcal{O}_{C_1}, \ldots, \mathcal{O}_{C_m}$ is a spherical collection.

Birational correspondences

$X, X' =$ birational varieties with trivial ω.

Examples

i) $X =$ CY, $\mathbb{P}^n \subset X$ with $\mathcal{N} = \mathcal{O}(-1)^{\oplus n+1}$.

$$\begin{array}{ccccccc} & & & \widetilde{X} & & & \\ \text{blow-up} & & \swarrow & & \searrow & & \text{blow-down} \\ \mathbb{P}^n & \hookrightarrow & X & & X' & \hookleftarrow & \mathbb{P}^n \end{array}$$

with exceptional divisor $E = \mathbb{P}^n \times \mathbb{P}^n$.

ii) $X =$ HK, $P := \mathbb{P}^n \subset X \Rightarrow \mathcal{N} = \Omega_P$.

$$\begin{array}{ccccccc} & & & \widetilde{X} & & & \\ \text{blow-up} & & \swarrow & & \searrow & & \text{blow-down} \\ P & \hookrightarrow & X & & X' & \hookleftarrow & P^* \end{array}$$

with exceptional divisor

$$E = \mathbb{P}(\Omega_P) = \{(\ell, H) \mid \ell \subset H\} \subset P \times P^*.$$

Remark Example ii) can be seen as a hyperplane section of example i).

$$
\begin{array}{ccccc}
 & \widetilde{X} & & & \\
\swarrow & & \searrow & & \\
X & & & X' & \\
\searrow & & \swarrow & & \\
 & \{0\} & & &
\end{array}
\quad \subset \quad
\begin{array}{ccccc}
 & \widetilde{\mathcal{X}} & & & \\
\swarrow & & \searrow & & \\
\mathcal{X} & & & \mathcal{X}' & \\
\searrow & & \swarrow & & \\
 & \mathbb{P}^1 & & &
\end{array}
$$

and $\mathcal{X} \cong \mathcal{X}'$ over $\mathbb{P}^1 \setminus \{0\}$.

General case

Batyrev, Kontsevich (1999) $X, X' =$ birational CY $\Rightarrow h^{p,q}(X) \stackrel{(*)}{=} h^{p,q}(X')$, $b_i(X) = b_i(X')$.

H. (1999) Let $X, X' =$ birational HK. Then there exists a diagram as above. In particular, X, X' are deformation equivalent. ($\Rightarrow$ $(*)$.)

Conjecture X, X' birational, K-equivalent (e.g. both ω trivial) $\Rightarrow \mathbf{D^b}(X) \cong \mathbf{D^b}(X')$.

Question $\mathbf{D^b}(X) \cong \mathbf{D^b}(X') \overset{?}{\Rightarrow} h^{p,q}(X) = h^{p,q}(X')$?

Bondal, Orlov (1995) In Example i): $\mathcal{O}_{\widetilde{X}} \in \mathrm{D^b}(X \times X')$ induces $\mathbf{D^b}(X) \cong \mathbf{D^b}(X')$.

Kawamata, Namikawa (2002) In Example ii): $\mathcal{O}_{\widetilde{X} \cup (P \times P^*)} \in \mathbf{D^b}(X \times X')$ induces $\mathbf{D^b}(X) \cong \mathbf{D^b}(X')$. (Note that $\mathcal{O}_{\widetilde{X}}$ does not.)

Remark There is a commutative diagram

$$\begin{array}{ccc} \mathbf{D^b}(X) & \xrightarrow{i_*} & \mathbf{D^b}(\mathcal{X}) \\ \Phi\downarrow & & \downarrow\Phi \\ \mathbf{D^b}(X') & \xrightarrow{i_*} & \mathbf{D^b}(\mathcal{X}'). \end{array}$$

Bridgeland (2001) Conjecture OK in dimension three.

Consider X, X' as in Example i) $\rightsquigarrow$

$$\Phi^2 : \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(X') \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(X)$$

both equivalences given by $\mathcal{O}_{\widetilde{X}}(E)$.

Question Does this define a new autoequivalence?

Kawamata In dimension three:

$$\Phi^2 = T_{\mathcal{O}_C(-1)}.$$

In higher dimensions Φ^2 is not a spherical twist, i.e. get genuinely new autoequivalence.

The other ones

Kawamata (2002) If X, X' are of general type, then

$$\text{K} - \text{equivalent} \Rightarrow \text{D} - \text{equivalent}.$$

More precisely, for any FM equivalence

$$\Phi_{\mathcal{P}} : \mathbf{D}^{\mathbf{b}}(X) \cong \mathbf{D}^{\mathbf{b}}(X')$$

the support supp($\mathcal{P}$) induces a birational correspondence

$$\begin{array}{ccc} & Z & \\ {}^{q}\swarrow & & \searrow^{p} \\ X & & X' \end{array}$$

with $q^*\omega_X \cong p^*\omega_{X'}$.

Low dimensions

Already $X, X' =$ curves. Then

$$X \cong X' \Leftrightarrow \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X').$$

Bridgeland, Maciocia (2001) $X, X' =$ surfaces. Then

$$X \cong X' \Leftrightarrow \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$$

except possibly if $X =$ K3, abelian, or relatively minimal elliptic.

Moreover $\mathrm{D}^{\mathrm{b}}(X)$ determines X also if X is Enriques or hyperelliptic.

Remark We don't know how to reduce to minimal varieties in dimension > 2.