GEOMETRIC EVOLUTION EQUATIONS

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CONTENTS

0. OVERVIEW

We consider flow equations that deform manifolds according to their curvature. In the lectures, I will focus on two types of such flow equations.

If $X_0 : M^n \to \mathbb{R}^{n+1}$ is an embedding of an n-dimensional manifold, we can define principal curvatures $(\lambda_i)_{1\le i\le n}$ and a normal vector ν. We deform the embedding vector according to

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot,0) = X_0, \end{cases}$$

where F is a symmetric function of the principal curvatures, e. g. the mean curvature $H = \lambda_1 + \cdots + \lambda_n$. In this way, we obtain a family $X(\cdot,t)$ of embeddings and study their behavior near singularities and for large times. We consider hypersurfaces that contract to a point in finite time and, after appropriate rescaling, to a round sphere. Graphical solutions are shown to exist for all times.

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We will also consider Ricci flow, where the metric $g(t)$ of a Riemannian manifold M evolves according to

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}\,(g(t)), \\ g(0) = g_0. \end{cases}$$

Manifolds that are initially positively curved in an appropriate sense converge to space-forms after rescaling. For other manifolds, solutions exist for all times.

Nowadays classical results were obtained by G. Huisken [27] for mean curvature flow and by R. Hamilton [26] for Ricci flow.

During the last years, geometric evolution equations have been used to study geometric questions like isoperimetric inequalities, the Schönfliess conjecture, the Poincaré conjecture, Thurston's geometrization conjecture, the 1/4-pinching theorem, or Yau's uniformization conjecture.

In this course, I'll give an introduction to both types of flow equations. We will learn to compute evolution equations, study manifolds that become round in finite time and those that evolve smoothly for all times. At the end, I'll indicate briefly, how evolution equations and surgery can be used to address geometric problems.

Remark 0.1.

(i) We will use geometric flow equations as a tool to canonically deform a manifold into a manifold with nicer properties.
(ii) The flow equations considered share many properties with the heat equation. In particular, they tend to balance differences, e. g. of the curvature, on the manifold.
(iii) In order to control the behavior of the flow, we will look for properties of the manifold that are preserved under the flow.
(iv) For precise control on the behavior of the evolving manifold, we will look for quantities that are monotone and have geometric significance, i. e. their boundedness implies geometric properties of the evolving manifold. Often these quantities are not scaling invariant. Hence bounding them "brakes the scaling invariance".

1. Differential Geometry of Submanifolds

We will only consider hypersurfaces in Euclidean space.

We use $X = X(x,\,t) = (X^\alpha)_{1\le\alpha\le n}$ to denote the time-dependent embedding vector of a manifold M^n into $\mathbb{R}^{n+1}$ and $\frac{d}{dt}X = \dot{X}$ for its total time derivative. Set $M_t := X(M,\,t) \subset \mathbb{R}^{n+1}$. We will often identify an embedded manifold with its image. We will assume that X is smooth. Assume furthermore that M^n is smooth, orientable, connected, complete and $\partial M^n = \emptyset$. We choose $\nu = \nu(x) = (\nu^\alpha)_{1\le\alpha\le n+1}$ to be the outer unit normal vector to M_t at $x \in M_t$. The embedding $X(\cdot,\,t)$ induces at each point on M_t a metric $(g_{ij})_{1\le i,\,j\le n}$ and a second fundamental form $(h_{ij})_{1\le i,\,j\le n}$. Let (g^{ij}) denote the inverse of (g_{ij}). These tensors are symmetric. The principal curvatures $(\lambda_i)_{1\le i\le n}$ are the eigenvalues of the second fundamental form with respect to that metric. That is, at $p \in M$, for each principal curvature λ_i, there exists $0 \ne \xi \in T_pM \cong \mathbb{R}^n$ such that

$$\lambda_i \sum_{l=1}^n g_{kl}\xi^l = \sum_{l=1}^n h_{kl}\xi^l \text{ or, equivalently, } \lambda_i\xi^l = \sum_{k,r=1}^n g^{lk}h_{kr}\xi^r.$$

As usual, eigenvalues are listed according to their multiplicity. A surface is called strictly convex, if all principal curvatures are strictly positive. The inverse of the second fundamental form is denoted by $\left(\tilde{h}^{ij}\right)_{1\le i,\,j\le n}$.

Latin indices range from 1 to n and refer to geometric quantities on the surface, Greek indices range from 1 to $n+1$ and refer to components in the ambient space $\mathbb{R}^{n+1}$. In $\mathbb{R}^{n+1}$, we will always choose Euclidean coordinates. We use the Einstein summation convention for repeated upper and lower indices. Latin indices are raised and lowered with respect to the induced metric or its inverse $\left(g^{ij}\right)$, for Greek indices we use the flat metric $(\overline{g}_{\alpha\beta})_{1\le\alpha,\beta\le n+1} = (\delta_{\alpha\beta})_{1\le\alpha,\beta\le n+1}$ of $\mathbb{R}^{n+1}$. So the defining equation for the principal curvatures becomes $\lambda_i g_{kl}\xi^l = h_{kl}\xi^l$.

Denoting by $\langle\cdot,\,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^{n+1}$, we have

$$g_{ij} = \langle X_{,\,i},\, X_{,\,j}\rangle = X^\alpha_{,\,i}\delta_{\alpha\beta}X^\beta_{,\,j},$$

where we used indices, preceded by commas, to denote partial derivatives. We write indices, preceded by semi-colons, e. g. $h_{ij;\,k}$ or $v_{;k}$, to indicate covariant differentiation with respect to the induced metric. Later, we will also drop the semi-colons, if the meaning is clear from the context. We set $X^\alpha_{;i}\equiv X^\alpha_{,i}$ and

$$X^\alpha_{;\,ij} = X^\alpha_{,\,ij} - \Gamma^k_{ij}X^\alpha_{,\,k}, \tag{1.1}$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{kl}(g_{il,\,j} + g_{jl,\,i} - g_{ij,\,l})$$

are the Christoffel symbols of the metric (g_{ij}). So $X^\alpha_{;ij}$ becomes a tensor.

The Gauß formula relates covariant derivatives of the position vector to the second fundamental form and the normal vector

$$X^\alpha_{;\,ij} = -h_{ij}\nu^\alpha. \tag{1.2}$$

The Weingarten equation allows to compute derivatives of the normal vector

$$\nu^\alpha_{;\,i} = h^k_i X^\alpha_{;\,k}. \tag{1.3}$$

We can use the Gauß formula (1.2) or the Weingarten equation (1.3) to compute the second fundamental form.

Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature $H = \lambda_1 + \ldots + \lambda_n$, the square of the norm of the second fundamental form $|A|^2 = \lambda_1^2 + \ldots + \lambda_n^2$, $\operatorname{tr} A^k = \lambda_1^k + \ldots + \lambda_n^k$, and the Gauß curvature $K = \lambda_1\cdot\ldots\cdot\lambda_n$. It is often convenient to choose coordinate systems such that, at a fixed point, the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and (h_{ij}) is diagonal, $(h_{ij}) = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$, e. g.

$$\sum \lambda_k h^2_{ij;k} = \sum_{i,\,j,\,k=1}^n \lambda_k h^2_{ij;k} = h^{kl}h^i_{j;\,k}h^j_{i;\,l} = h_{rs}h_{ij;\,k}h_{ab;\,l}g^{ia}g^{jb}g^{rk}g^{sl}.$$

Whenever we use this notation, we will also assume that we have fixed such a coordinate system.

A normal velocity F can be considered as a function of $(\lambda_1,\ldots,\lambda_n)$ or $(h_{ij},\,g_{ij})$. If $F(\lambda_i)$ is symmetric and smooth, then $F(h_{ij}, g_{ij})$ is also smooth [22, Theorem 2.1.20]. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,\,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}$. Note that in coordinate systems with diagonal h_{ij} and $g_{ij} = \delta_{ij}$ as mentioned above, F^{ij} is diagonal. For $F = |A|^2$, we have $F^{ij} = 2h^{ij} = 2\lambda_i g^{ij}$, and for $F = -K^{-1}$, we have $F^{ij} = K^{-1}\tilde{h}^{ij} = K^{-1}\lambda_i^{-1}g^{ij}$.

The Gauß equation expresses the Riemannian curvature tensor of the surface in terms of the second fundamental form

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}. \tag{1.4}$$

As we use only Euclidean coordinate systems in $\mathbb{R}^3$, $h_{ij;\,k}$ is symmetric according to the Codazzi equations.

The Ricci identity allows to interchange covariant derivatives. We will use it for the second fundamental form

$$h_{ik;\,lj} = h_{ik;\,jl} + h_k^a R_{ailj} + h_i^a R_{aklj}. \tag{1.5}$$

For tensors A and B, $A_{ij} \ge B_{ij}$ means that $(A_{ij} - B_{ij})$ is positive definite.

Finally, we use c to denote universal, estimated constants.

1.1. Graphical Submanifolds.

Lemma 1.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then* graph u *is a submanifold in $\mathbb{R}^{n+1}$. The metric g_{ij}, the lower unit normal vector ν, the second fundamental form h_{ij}, the mean curvature H, and the Gauß curvature K are given by*

$$\begin{aligned} g_{ij} &= \delta_{ij} + u_i u_j, \\ g^{ij} &= \delta^{ij} - \frac{u^i u^j}{1+|Du|^2}, \\ \nu &= \frac{((u_i), -1)}{\sqrt{1+|Du|^2}} \equiv \frac{((u_i), -1)}{v}, \\ h_{ij} &= \frac{u_{ij}}{\sqrt{1+|Du|^2}} \equiv \frac{u_{ij}}{v}, \\ H &= \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right), \end{aligned}$$

and

$$K = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+2}{2}}},$$

where $u_i \equiv \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. Note that in Euclidean space, we don't need to distinguish between Du and ∇u.

Proof.

(i) We use the embedding vector $X(x) := (x, u(x))$, $X : \mathbb{R}^n \to \mathbb{R}^{n+1}$. The induced metric is the pull-back of the metric in Euclidean $\mathbb{R}^{n+1}$, $g := X^* g_{\mathbb{R}^{n+1}_{\text{Eucl.}}}$. We have $X_{,i} = (e_i, u_i)$. Hence

$$g_{ij} = X^\alpha_{,i}\delta_{\alpha\beta}X^\beta_{,j} = \langle X_{,i}, X_{,j}\rangle = \langle (e_i, u_i), (e_j, u_j)\rangle = \delta_{ij} + u_i u_j.$$

(ii) It is easy to check, that g^{ij} is the inverse of g_{ij}. Note that $u^i := \delta^{ij}u_j$, i.e., we lift the index with respect to the flat metric. It is convenient to choose a coordinate system such that $u_i = 0$ for $i < n$.

(iii) The vectors $X_{,i} = (e_i, u_i)$ are tangent to graph u. The vector $((-u_i), 1) \equiv (-Du, 1)$ is orthogonal to these vectors, hence, up to normalization, a unit normal vector.

(iv) We combine (1.1), (1.2) and compute the scalar product with ν to get

$$\begin{aligned} h_{ij} &= -\langle X_{;ij}, \nu\rangle = -\langle X_{,ij} - \Gamma^k_{ij} X_{,k}, \nu\rangle = -\langle X_{,ij}, \nu\rangle \\ &= -\left\langle (0, u_{ij}), \frac{((u_i), -1)}{v}\right\rangle = \frac{u_{ij}}{v}. \end{aligned}$$

(v) We obtain

$$\begin{aligned} H = \sum_{i=1}^n \lambda_i = g^{ij} h_{ij} &= \left(\delta^{ij} - \frac{u^i u^j}{1+|Du|^2}\right) \frac{u_{ij}}{\sqrt{1+|Du|^2}} \\ &= \frac{\delta^{ij} u_{ij}}{\sqrt{1+|Du|^2}} - \frac{u^i u^j u_{ij}}{(1+|Du|^2)^{3/2}} \\ &= \frac{\Delta u}{\sqrt{1+|Du|^2}} - \frac{u^i u^j u_{ij}}{(1+|Du|^2)^{3/2}} \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) &= \sum_{i=1}^n \frac{\partial}{\partial x^i} \frac{u_i}{\sqrt{1+|Du|^2}} \\ &= \sum_{i=1}^n \frac{u_{ii}}{\sqrt{1+|Du|^2}} - \sum_{i,j=1}^n \frac{u_i u_j u_{ji}}{(1+|Du|^2)^{3/2}} \\ &= H. \end{aligned}$$

(vi) From the defining equation for the principal curvatures, we obtain

$$\begin{aligned} K = \prod_{i=1}^n \lambda_i &= \det\left(g^{ij} h_{jk}\right) = \det g^{ij} \cdot \det h_{ij} = \frac{\det h_{ij}}{\det g_{ij}} \\ &= \frac{v^{-n} \det u_{ij}}{v^2} = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+2}{2}}}. \end{aligned}$$

□

Exercise 1.2 (Spheres)**.** The lower part of a sphere of radius R is locally given as graph u with $u : B_R(0) \to \mathbb{R}$ defined by $u(x) := -\sqrt{R^2 - |x|^2}$. Compute explicitly for that example all the quantities mentioned in Lemma 1.1 and the principal curvatures.

Exercise 1.3. Give a geometric definition of the (principal) curvature of a curve in $\mathbb{R}^2$ in terms of a circle approximating that curve in an optimal way.

Use the min-max characterization of eigenvalues to extend that geometric definition to n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$.

Exercise 1.4 (Rotationally symmetric graphs)**.**
Assume that the function $u : \mathbb{R}^n \to \mathbb{R}$ is smooth and $u(x) = u(y)$, if $|x| = |y|$. Then $u(x) = f(|x|)$ for some $f : \mathbb{R}_+ \to \mathbb{R}$. Compute once again all the geometric quantities mentioned in Lemma 1.1.

2. Evolving Submanifolds

2.1. **General Definition.** We will only consider the evolution of manifolds of dimension n embedded into $\mathbb{R}^{n+1}$, i. e. the evolution of hypersurfaces in Euclidean space. (Mean curvature flow is also considered for manifolds of arbitrary codimension. Another generalization is to study flow equations of hypersurfaces immersed into a (Riemannian or Lorentzian) manifold.)

Definition 2.1. Let M^n denote an orientable manifold of dimension n. Let $X(\cdot,t):M^n\to\mathbb{R}^{n+1}$, $0\le t\le T\le\infty$, be a smooth family of smooth embeddings. Let ν denote one choice of the normal vector field along $X(M^n,t)$. Then $M_t:=X(M^n,t)$ is said to move with normal velocity F, if

$$\frac{d}{dt}X=-F\nu\quad\text{in } M^n\times[0,T).$$

In codimension 1, we often don't need to assume that M^n is orientable.

Remark 2.2. Let $X:M^n\to N^{n+1}$ be a C^2-immersion and $H_1(N;\mathbb{Z}/2\mathbb{Z})=0$. Assume that X is proper, $X^{-1}(\partial N)=\partial M$, and X is transverse to ∂N. Then $N\setminus f(M)$ is not connected [19]. Hence, if M^n is closed and embedded in $\mathbb{R}^{n+1}$, M^n is orientable.

In the following we will often identify an embedded submanifold and its image under the embedding.

2.2. **Evolution of graphs.**

Lemma 2.3. *Let $u:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be a smooth function such that* graph u *evolves according to $\frac{d}{dt}X=-F\nu$. Then*

$$\dot u=\sqrt{1+|Du|^2}\cdot F.$$

Proof. Beware of assuming that considering the $n+1$-st component in the evolution equation $\frac{d}{dt}X=-F\nu$ were equal to $\dot u$ as a hypersurface evolving according to $\frac{d}{dt}X=-F\nu$ does not only move in vertical direction but also in horizontal direction.

Let p denote a point on the abstract manifold embedded via X into $\mathbb{R}^{n+1}$. As our embeddings are graphical, we see that

$$X(p,t)=(x(p,t),u(x(p,t),t)).$$

We consider the scalar product of both sides of the evolution equation with ν and obtain

$$F=\langle F\nu,\nu\rangle=\left\langle-\frac{d}{dt}X,\nu\right\rangle=-\left\langle\left(\left(\dot x^k\right),u_i\dot x^i+\dot u\right),\frac{((u_i),-1)}{\sqrt{1+|Du|^2}}\right\rangle=\frac{\dot u}{\sqrt{1+|Du|^2}}.$$

□

Corollary 2.4. *Let $u:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be a smooth function such that* graph u *solves mean curvature flow $\frac{d}{dt}X=-H\nu$. Then*

$$\dot u=\sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Exercise 2.5 (Rotationally symmetric translating solutions)**.** Let $u := \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be rotationally symmetric. Assume that graph u is a translating solution to mean curvature flow $\frac{d}{dt}X = -H\nu$, i. e. a solution such that $\dot{u}$ is constant.

Why does it suffice to consider the case $\dot{u} = 1$?

Similar to Exercise 1.4, derive an ordinary differential equation for translating rotationally symmetric solutions to mean curvature flow.

Remark 2.6. Consider a physical system consisting of a domain $\Omega \subset \mathbb{R}^3$. Assume that the energy of the system is proportional to the surface area of $\partial\Omega$. Then the L^2-gradient flow for the area is mean curvature flow. We check that in a model case for graphical solutions in Lemma 2.7.

Lemma 2.7. *Let $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be smooth. Assume that $u(x, 0) \equiv 0$ for $|x| \ge R$. Then the surface area is maximally reduced among all normal velocities F with given L^2-norm, if the normal velocity of* graph u *is given by H, i. e. if $\dot{u} = \sqrt{1 + |Du|^2}\, H$.*

Proof. The area of graph $u(\cdot, t)|_{B_R}$ is given by

$$A(t) = \int\limits_{B_R} \sqrt{1 + |Du|^2}\, dx.$$

Define the induced area element $d\mu$ by $d\mu := \sqrt{1 + |Du|^2}\, dx$. We obtain using integration by parts

$$\begin{aligned}
\frac{d}{dt}A(t)\bigg|_{t=0} &= \int\limits_{B_R} \frac{d}{dt}\sqrt{1+|Du|^2}\, dx\,\Bigg|_{t=0} = \int\limits_{B_R(0)} \frac{1}{\sqrt{1+|Du|^2}}\langle Du, D\dot{u}\rangle\,\Bigg|_{t=0} \\
&= -\int\limits_{B_R} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\frac{\dot{u}}{v}\cdot v\, dx\,\Bigg|_{t=0} = -\int\limits_{B_R} H\, F\, d\mu\,\Bigg|_{t=0} \\
&\ge -\left(\int\limits_{B_R} H^2\, d\mu\right)^{1/2}\left(\int\limits_{B_R} F^2\, d\mu\right)^{1/2}\Bigg|_{t=0}.
\end{aligned}$$

Here, we have used Hölder's inequality $\|ab\|_{L^1} \le \|a\|_{L^2} \cdot \|b\|_{L^2}$. There, we get equality precisely if a and b differ only by a multiplicative constant. Hence the surface area is reduced most efficiently among all normal velocities F with $\|F\|_{L^2} = \|H\|_{L^2}$, if we choose $F = H$. In this sense, mean curvature flow is the L^2-gradient flow for the area integral. $\square$

2.3. **Examples.**

Lemma 2.8. *Consider mean curvature flow, i. e. the evolution equation $\frac{d}{dt}X = -H\nu$, with $M_0 = \partial B_R(0)$. Then a smooth solution exists for $0 \le t < T := \frac{1}{2n}R^2$ and is given by $M_t = \partial B_{r(t)}(0)$ with $r(t) = \sqrt{2n(T-t)} = \sqrt{R^2 - 2nt}$.*

Proof. The mean curvature of a sphere of radius $r(t)$ is given by $H = \frac{n}{r(t)}$. Hence we obtain a solution to mean curvature flow, if $r(t)$ fulfills

$$\dot{r}(t) = \frac{-n}{r(t)}.$$

A solution to this ordinary differential equation is given by $r(t) = \sqrt{2n(T-t)}$.

(The theory of partial differential equations implies that this solution is actually unique and hence no solutions exist that are not spherical.) $\square$

Exercise 2.9. Find a solution to mean curvature flow with $M_0 = \partial B_R(0) \times \mathbb{R}^k \subset \mathbb{R}^l \times \mathbb{R}^k$. This includes in particular cylinders. Note that for $k > 1$, it is not obvious, whether these solutions are unique.

Exercise 2.10. Find solutions for $\frac{d}{dt}X = -|A|^2\nu$, $\frac{d}{dt}X = -K\nu$, $\frac{d}{dt}X = \frac{1}{H}\nu$, and $\frac{d}{dt}X = \frac{1}{K}\nu$ if $M_0 = \partial B_R(0) \subset \mathbb{R}^{n+1}$, especially for $n = 2$.

Remark 2.11 (Level-set flow)**.** Let M_t be a family of smooth embedded hypersurfaces in $\mathbb{R}^{n+1}$ that move according to $\frac{d}{dt}X = -F\nu$ with $F > 0$. Impose the global assumption that each point $x \in \mathbb{R}^{n+1}$ belongs to at most one hypersurface M_t. Then we can (at least locally) define a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ by setting $u(x) = t$, if $x \in M_t$. That is $u(x)$ is the time, at which the hypersurface passes the point x. We obtain the equation $F \cdot |Du| = 1$.

If $F < 0$, we get $F \cdot |Du| = -1$.

This formulation is used to describe weak solutions, where singularities in the classical formulation occur. See for example [28], where the inverse mean curvature flow $F = -\frac{1}{H}$ is considered to prove the Riemannian Penrose inequality. Note that $H = \operatorname{div}\left(\frac{Du}{|Du|}\right)$ as the outer unit normal vector to a closed expanding hypersurface $M_t = \{u = t\}$ is given by $\frac{Du}{|Du|}$. According to (1.3), the divergence of the unit normal yields the mean curvature as the derivative of the unit normal in the direction of the unit normal vanishes. Hence the evolution equation $\frac{d}{dt}X = \frac{1}{H}\nu$ can be rewritten as

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du|.$$

Mean curvature flow can be rewritten as $|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = -1$.

Exercise 2.12. Verify the formula for the mean curvature in the level-set formulation. Compute level-set solutions to the flow equations $\frac{d}{dt}X = -H\nu$ and $\frac{d}{dt}X = \frac{1}{H}\nu$, where u depends only on $|x|$, i. e. the hypersurfaces M_t are spheres centered at the origin. Compare the result to your earlier computations.

We will use the level-set formulation to study a less trivial solution to mean curvature flow which can be written down in closed form.

Exercise 2.13 (Paper-clip solution)**.** Let $v \neq 0$. Consider the set

$$M_t := \left\{(x, y) \in \mathbb{R}^2 : e^{v^2 t} \cosh(vy) = \cos(vx)\right\}.$$

Show that M_t solves mean curvature flow. Describe the shape of M_t for $t \to -\infty$ and for $t \uparrow 0$ (after appropriate rescaling).

Compare this to Theorem 5.1.

Note that you may also rewrite solutions equivalently (on an appropriate domain) as

$$y_\pm := \frac{1}{v} \log\left(\cos(vx) \pm \sqrt{\cos^2(vx) - e^{2v^2 t}}\right) - vt.$$

Hint: You should obtain $t_x = u_x = -\frac{\sin(vx)}{v\cos(vx)}$ and $u_y = -\frac{\sinh(vy)}{v\cosh(vy)}$.

2.4. **Short-time existence and avoidance principle.** In the case of closed initial hypersurfaces, short-time existence is guaranteed by the following

Theorem 2.14 (Short-time existence)**.** *Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be an embedding describing a smooth closed hypersurface. Let $F = F(\lambda_i)$ be smooth, symmetric, and $\frac{\partial F}{\partial \lambda_i} > 0$ everywhere on $X(M^n)$ for all i. Then the initial value problem*

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot, 0) = X_0 \end{cases}$$

has a smooth solution on some (short) time interval $[0, T)$, $T > 0$.

Idea of Proof. Represent solutions locally as graphs in a tubular neighborhood of $X_0(M^n)$. Then $\frac{\partial F}{\partial \lambda_i} > 0$ ensures that the evolution equation for the height function in this coordinate system is strictly parabolic. Linear theory and the implicit function theorem guarantee that there exists a solution on a short time interval.

For details see [29, Theorem 3.1]. □

Exercise 2.15. Check, for which initial data the conditions in Theorem 2.14 are fulfilled if $F = H$, K, $|A|^2$, $-1/H$, $-1/K$.

Find examples of closed hypersurfaces such that

(i) $H > 0$,
(ii) $K > 0$,
(iii) H is not positive everywhere,
(iv) $H > 0$, but K changes sign.

Show that on every smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$, there is a point, where M^n is strictly convex, i. e. $\lambda_i > 0$ is fulfilled for every i.

On the other hand, starting with a closed hypersurface gives rise to solutions that exist at most on a finite time interval. This is a consequence of the following

Theorem 2.16 (Avoidance principle)**.** *Let $F = F(\lambda_i)$ be smooth and symmetric. Let M_t^1 and $M_t^2 \subset \mathbb{R}^{n+1}$ be two embedded closed hypersurfaces and smooth solutions to a strictly parabolic flow equation $\frac{d}{dt}X = -F\nu$, i. e. $\frac{\partial F}{\partial \lambda_i} > 0$ during the flow. Assume that F, considered as a function of (D^2u, Du) for graphs, is elliptic on a set, which is convex and independent of Du. If M_0^1 and M_0^2 are disjoint, M_t^1 and M_t^2 can only touch if the respective normal vectors fulfill $\nu^1 = -\nu^2$ there. Hence, if M_0^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_0^2$, then M_t^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_t^2$ unless the hypersurfaces touch each other in a point with opposite normals.*

The technical condition on the convexity of the domain, where F, considered as a function of (D^2u, Du), is convex, is technical and always fulfilled for the evolution equations considered here (besides for the inverse mean curvature flow). It can be relaxed, but makes the proof less transparent. Only for $F = -\frac{1}{H}$, a separate argument is needed. (It suffices to choose coordinates such that $|Du| \ll 1$. Then interpolation does not destroy positivity of the denominator. The technical details are left as an exercise.)

The normal velocity F is a symmetric function of the principal curvatures. Thus it is well-defined, as the principal curvatures are defined only up to permutations.

We have considered F as a function of the principal curvatures. Writing an evolving hypersurface locally as graph u, we also wish to express F in terms of

(D^2u, Du). We continue to call this function F. In the cases considered here, it is clear from the explicit expressions, that F is also a smooth function of (D^2u, Du). In general, this is a theorem [22, Theorem 2.1.20].

A similar statement is true for the condition $\frac{\partial F}{\partial \lambda_i} > 0$ and the ellipticity of $F\left(D^2, Du\right)$, i.e. $0 < \frac{F(r,p)}{r_{ij}} < \infty$ in the sense of matrices. Once again, it can be checked by direct computations for the normal velocities considered here, that these two statements are equivalent.

Proof of Theorem 2.16. Otherwise there would be some $t_0 > 0$ such that $M^2_{t_0}$ touches $M^1_{t_0}$ at some point $p \in \mathbb{R}^{n+1}$ with normal vectors $\nu^1 = \nu^2$ at p. Writing M^i_t locally as graph u^i over the common tangent hyperplane $T_pM^i_{t_0} \subset \mathbb{R}^{n+1}$, we see that the functions u^i fulfill $\dot{u}^i = F\left(D^2u^i, Du^i\right)$ for some strictly elliptic differential operator F corresponding to the normal velocity F. We may assume that $u^1 > u^2$ for $t < t_0$. The evolution equation for the difference $w := u_1 - u_2$ fulfills $w > 0$ for $t < t_0$ locally in space-time and $w(0, t_0)$, if we have $p = (0, 0)$ in our coordinate system. The evolution equation for w can be computed as follows

$$\begin{aligned}\dot{w} &= \dot{u}^1 - \dot{u}^2 = F\left(D^2u^1, Du^1\right) - F\left(D^2u^2, Du^2\right)\\ &= \int\limits_0^1 \frac{d}{d\tau} F\left(\tau D^2u^1 + (1-\tau)D^2u^2, \tau Du^1 + (1-\tau)Du^2\right)\, d\tau\\ &= \int\limits_0^1 \frac{\partial F}{\partial r_{ij}}(\ldots)\, d\tau \cdot \left(u^1 - u^2\right)_{ij} + \int\limits_0^1 \frac{\partial F}{\partial p_i}(\ldots)\, d\tau \cdot \left(u^1 - u^2\right)_i\\ &\equiv a^{ij} w_{ij} + b^i w_i.\end{aligned}$$

Hence we can apply the parabolic Harnack inequality or the strong parabolic maximum principle and see that it is impossible that $w(x, t) > 0$ for small $|x|$ and $t < t_0$, but $w(0, t_0) = 0$. Hence M^1_t can't touch M^2_t in a point, where $\nu^1 = \nu^2$. The theorem follows. $\square$

Exercise 2.17. Show that the normal velocities as considered in Exercise 2.15 can be represented (in an appropriate domain) as smooth functions of (D^2u, Du) for hypersurfaces that are locally represented as graph u.

Denote by Γ_F the set of $(\lambda_i) \subset \mathbb{R}^n$ such that $\frac{\partial F}{\partial \lambda_i} > 0$. Show that this set is a convex cone. Prove that F as a function (D^2u, Du) is strictly elliptic precisely if the principal curvatures corresponding to (D^2u, Du) lie in Γ_F.

Corollary 2.18. *Let M_0 be a smooth closed embedded hypersurface in $\mathbb{R}^{n+1}$. Then a smooth solution M_t to $\frac{d}{dt}X = -H\nu$ can only exist on some finite time interval $[0, T)$, $T < \infty$.*

Proof. Choose a large sphere that encloses M_0. According to Lemma 2.8, that sphere shrinks to a point in finite time. Thus the solution M_t can exist smoothly at most up to that time. $\square$

Exercise 2.19. Deduce similar corollaries for the normal velocities in Exercise 2.15. You may use Exercise 2.10.

Consider T maximal such that a smooth solution M_t as in Corollary 2.18 exists on $[0, T)$. Then the embedding vector X is uniformly bounded according to

Theorem 2.16. Then some spatial derivative of the embedding $X(\cdot, t)$ has to become unbounded as $t \uparrow T$. For otherwise we could apply Arzelà-Ascoli and obtain a smooth limiting hypersurface M_T such that M_t converges smoothly to M_T as $t \uparrow T$. This, however, is impossibly, as Theorem 2.14 would allow to restart the flow from M_T. In this way, we could extend the flow smoothly all the way up to $T + \varepsilon$ for some $\varepsilon > 0$, contradicting the maximality of T.

It can often be shown that extending a solution beyond T is possible provided that $\|X(\cdot, t)\|_{C^2}$ is uniformly bounded. For mean curvature flow, this follows from explicit estimates. For other normal velocities, additional assumptions (the principal curvatures stay in a region, where F has nice properties) and Krylov-Safonov-estimates can imply such a result.

3. Evolution Equations for Submanifolds

In this chapter, we will compute evolution equations of geometric quantities, see e. g. [27, 29, 39].

For a family M_t of hypersurfaces solving the evolution equation

$$\frac{d}{dt} X = -F\nu \tag{3.1}$$

with $F = F(\lambda_i)$, where F is a smooth symmetric function, we have the following evolution equations.

Lemma 3.1. *The metric g_{ij} evolves according to*

$$\frac{d}{dt} g_{ij} = -2Fh_{ij}. \tag{3.2}$$

Proof. By definition, $g_{ij} = \langle X_{,i}, X_{,j}\rangle = X^\alpha_{,i}\delta_{\alpha\beta}X^\beta_{,j}$. We differentiate with respect to time. Derivatives of $\delta_{\alpha\beta}$ vanish. The term $X^\alpha_{,i}$ involves only partial derivatives. We obtain

$$\frac{d}{dt} g_{ij} = \left(\dot{X}^\alpha\right)_{,i} \delta_{\alpha\beta}X^\beta_{,j} + X^\alpha_{,i}\delta_{\alpha\beta}\left(\dot{X}^\beta\right)_{,j}$$

(we may exchange partial spatial and time derivatives)

$$= (-F\nu^\alpha)_{,i}\delta_{\alpha\beta}X^\beta_{,j} + X^\alpha_{,i}\delta_{\alpha\beta}(-F\nu\beta)_{,j}$$

(in view of the evolution equation $\frac{d}{dt}X = -F\nu$)

$$= -F\nu^\alpha_{;i}\delta_{\alpha\beta}X^\beta_{,j} - X^\alpha_{,i}\delta_{\alpha\beta}F\nu_{;j}$$

(terms involving derivatives of F vanish as ν and $X^\alpha_{,i}$ are orthogonal to each other; as the background metric $\overline{g}_{\alpha\beta} = \delta_{\alpha\beta}$ is flat, covariant and partial derivatives of ν coincide)

$$= -Fh^k_i X^\alpha_{,k}\delta_{\alpha\beta}X^\beta_{,j} - FX^\alpha_{,i}\delta_{\alpha\beta}h^k_j X^\beta_{,k}$$

(in view of the Weingarten equation (1.3))

$$= -Fh^k_i g_{kj} - Fg_{ik}h^k_j$$

(by the definition of the metric)

$$= -2Fh_{ij}$$

(by the definition of $h^i_j := h_{jk}g^{ki}$).

The lemma follows. $\square$

Corollary 3.2. *The evolution equation of the volume element $d\mu := \sqrt{\det g_{ij}}\,dx$ is given by*

$$\frac{d}{dt}d\mu = -FH\,d\mu. \tag{3.3}$$

Proof. Exercise. Recall the formulae for differentiating the determinant and the inverse of a matrix. $\square$

Lemma 3.3. *The unit normal ν evolves according to*

$$\frac{d}{dt}\nu^\alpha = g^{ij}F_{;\,i}X^\alpha_{;\,j}. \tag{3.4}$$

Proof. By definition, the unit normal vector ν has length one, $\langle\nu,\nu\rangle = 1 = \nu^\alpha\delta_{\alpha\beta}\nu^\beta$. Differentiating yields

$$0 = \dot\nu^\alpha\delta_{\alpha\beta}\nu^\beta.$$

Hence it suffices to show that the claimed equation is true if we take on both sides the scalar product with an arbitrary tangent vector. The vectors $X_{,i}$ (which we will also denote henceforth by X_i as there is no danger of confusion; we will also adopt this convention if partial and covariant derivatives of some quantity coincide) form a basis of the tangent plane at a fixed point. We differentiate the relation

$$0 = \langle\nu, X_i\rangle = \nu^\alpha\delta_{\alpha\beta}X^\beta_i$$

and obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i + \nu^\alpha\delta_{\alpha\beta}\frac{d}{dt}X^\beta_i \\ &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i + \nu^\alpha\delta_{\alpha\beta}\left(\frac{d}{dt}X^\beta\right)_i \\ &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i - \nu^\alpha\delta_{\alpha\beta}\left(F\nu^\beta\right)_i. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i &= \nu^\alpha\delta_{\alpha\beta}\nu^\beta F_i + F\nu^\alpha\delta_{\alpha\beta}\nu^\beta_i \\ &= F_i + F\tfrac{1}{2}\langle\nu,\nu\rangle_i = F_i \end{aligned}$$

and the lemma follows as taking the scalar product of the claimed evolution equation with X_k, i. e. multiplying it with $\delta_{\alpha\beta}X^\beta_k$, yields

$$\frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_k = g^{ij}F_iX^\alpha_j\delta_{\alpha\beta}X^\beta_k = g^{ij}F_ig_{jk} = \delta^i_kF_i = F_k.$$

$\square$

Lemma 3.4. *The second fundamental form h_{ij} evolves according to*

$$\frac{d}{dt}h_{ij} = F_{;\,ij} - Fh^k_ih_{kj}. \tag{3.5}$$

Proof. The Gauß formula (1.2) implies that $h_{ij} = -X^\alpha_{;ij}\nu_\alpha$. Differentiating yields

$$\begin{aligned}
\frac{d}{dt}h_{ij} &= -\frac{d}{dt}\langle X_{;ij},\nu\rangle\\
&= -\left\langle \frac{d}{dt}X_{;ij},\nu\right\rangle - \left\langle -h_{ij}\nu, \frac{d}{dt}\nu\right\rangle\\
&= -\left\langle \frac{d}{dt}X_{;ij},\nu\right\rangle + h_{ij}\left\langle \nu, \frac{d}{dt}\nu\right\rangle\\
&= -\left\langle \frac{d}{dt}X_{;ij},\nu\right\rangle\\
&= -\frac{d}{dt}\left(X^\alpha_{,ij} - \Gamma^k_{ij}X^\alpha_k\right)\nu_\alpha\\
&= -\left(\frac{d}{dt}X^\alpha\right)_{,ij}\nu_\alpha + \Gamma^k_{ij}\left(\frac{d}{dt}X^\alpha\right)_{,k}\nu_\alpha
\end{aligned}$$

(where no time derivatives of Γ^k_{ij} show up as $X^\alpha_i\nu_\alpha = 0$)

$$= (F\nu^\alpha)_{,ij}\nu_\alpha - \Gamma^k_{ij}(F\nu^\alpha)_{,k}\nu_\alpha$$

(in view of the evolution equation)

$$\begin{aligned}
&= F_{,ij}\nu^\alpha\nu_\alpha + F_{,i}\nu^\alpha_{,j}\nu_\alpha + F_{,j}\nu^\alpha_{,i}\nu_\alpha + F\nu^\alpha_{,ij}\nu_\alpha - \Gamma^k_{ij}F_{,k}\nu^\alpha\nu_\alpha - \Gamma^k_{ij}F\nu^\alpha_{,k}\nu_\alpha\\
&= F_{;ij} + F\nu^\alpha_{,ij}\nu_\alpha
\end{aligned}$$

as $F_{;ij} = F_{,ij} - \Gamma^k_{ij}F_{,k}$ and $\nu^\alpha_{,j}\nu_\alpha = \frac{1}{2}(\nu^\alpha\nu_\alpha)_j = 0$. It remains to show that $\nu^\alpha_{,ij}\nu_\alpha = -h^k_i h_{kj}$. We obtain

$$\nu^\alpha_{,ij}\nu_\alpha = \nu^\alpha_{;i,j}\nu_\alpha$$

(as $\nu^\alpha_i = \nu^\alpha_{;i}$)

$$= \nu^a_{;ij}\nu_\alpha$$

($\nu^\alpha_{;ij} = (\nu^\alpha_{;i})_{,j} - \Gamma^k_{ij}\nu^\alpha_k$ and $0 = \nu^\alpha_k\nu_\alpha$)

$$= \left(h^k_i X^\alpha_k\right)_{;j}\nu_\alpha$$

(according to the Weingarten equation (1.3))

$$= h^k_i(-h_{kj}\nu^\alpha)\nu_\alpha$$

(due to the Gauß equation (1.2) and the orthogonality $X^\alpha_k\nu_\alpha = 0$)

$$= -h^k_i h_{kj}$$

as claimed. The Lemma follows. □

Lemma 3.5. *The normal velocity F evolves according to*

$$\frac{d}{dt}F - F^{ij}F_{;ij} = FF^{ij}h_i^k h_{kj}. \tag{3.6}$$

Proof. We have, see [37, Lemma 5.4], the proof of [22, Theorem 2.1.20], or check this explicitly for the normal velocity considered,

$$\frac{\partial F}{\partial g_{kl}} = -F^{il}h_i^k$$

and compute the evolution equation of the normal velocity F

$$\begin{aligned}\frac{d}{dt}F - F^{ij}F_{;ij} =& - F^{il}h_i^k\frac{d}{dt}g_{kl} + F^{ij}\frac{d}{dt}h_{ij} - F^{ij}F_{;\,ij}\\ =& FF^{ij}h_i^k h_{kj},\end{aligned}$$

where we used (3.2) and (3.5). □

We will need more explicit evolution equations for geometric quantities $\boxplus$ involving $\frac{d}{dt}\boxplus - F^{ij}\boxplus_{;ij}$.

Lemma 3.6. *The second fundamental form h_{ij} evolves according to*

$$\begin{aligned}\frac{d}{dt}h_{ij} - F^{kl}h_{ij;\,kl} =& F^{kl}h_k^a h_{al}\cdot h_{ij} - F^{kl}h_{kl}\cdot h_i^a h_{aj}\\ & - Fh_i^k h_{kj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}.\end{aligned} \tag{3.7}$$

Proof. Direct calculations yield

$$\begin{aligned}\frac{d}{dt}h_{ij} - F^{ij}h_{ij;\,kl} =& F_{;ij} - Fh_i^k h_{kj} - F^{ij}h_{ij;kl} && \text{by (3.5)}\\ =& F^{kl}h_{kl;\,ij} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}\\ & - Fh_i^k h_{kj} - F^{ij}h_{ij;\,kl}\\ =& F^{kl}h_{ik;\,lj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}\\ & - Fh_i^k h_{kj} - F^{ij}h_{ik;\,jl} && \text{by Codazzi}\\ =& F^{kl}\left(h_k^a R_{ailj} + h_i^a R_{aklj}\right) - Fh_i^k h_{kj}\\ & + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j} && \text{by (1.5)}\\ =& F^{kl}h_k^a h_{al}h_{ij} - F^{kl}h_k^a h_{aj}h_{il}\\ & + F^{kl}h_i^a h_{al}h_{kj} - F^{kl}h_i^a h_{aj}h_{kl}\\ & - Fh_i^k h_{kj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j} && \text{by (1.4)}\\ =& F^{kl}h_k^a h_{al}h_{ij} - F^{kl}h_i^a h_{aj}h_{kl}\\ & - Fh_i^k h_{kj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}.\end{aligned}$$

□

Remark 3.7. A direct consequence of (3.1) and (1.2) is

$$\frac{d}{dt}X^\alpha - F^{ij}X^\alpha_{;\,ij} = \left(F^{ij}h_{ij} - F\right)\nu^\alpha. \tag{3.8}$$

Hence

(3.9)
$$\frac{d}{dt}|X|^2 - F^{ij}\left(|X|^2\right)_{;ij} = 2\left(F^{ij}h_{ij} - F\right)\langle X,\,\nu\rangle - 2F^{ij}g_{ij},$$
(3.10)

Proof. We have
$$\begin{aligned}\frac{d}{dt}|X|^2 - F^{ij}\left(|X|^2\right)_{;ij} =& 2\left\langle X, \frac{d}{dt}X\right\rangle - 2F^{ij}\langle X_i, X_j\rangle - 2F^{ij}\langle X, X_{;ij}\rangle\\ =& 2\langle X, -F\nu\rangle - 2F^{ij}g_{ij} - 2F^{ij}\langle X, -h_{ij}\nu\rangle.\end{aligned}$$
□

Lemma 3.8. *The evolution equation for the unit normal* ν *is*

$$\frac{d}{dt}\nu^\alpha - F^{ij}\nu_{;ij} = F^{ij}h_i^k h_{kj}\cdot\nu^\alpha. \tag{3.11}$$

Proof. We compute
$$\begin{aligned}\frac{d}{dt}\nu^\alpha - F^{ij}\nu^\alpha_{;ij} =& g^{ij}F_{;\,i}X^\alpha_{;\,j} - F^{ij}\left(h_i^k X^\alpha_{;\,k}\right)_{;\,j} && \text{by (3.4) and (1.3)}\\ =& g^{ij}F^{kl}h_{kl;\,i}X^\alpha_{;\,j} - F^{ij}h^k_{i;\,j}X^\alpha_{;\,k} - F^{ij}h_i^k X^\alpha_{;\,kj}\\ =& F^{ij}h_i^k h_{kj}\nu^\alpha && \text{by (1.2).}\end{aligned}$$
□

Lemma 3.9. *The evolution equation for the scalar product* $\langle X,\nu\rangle$ *is*

$$\frac{d}{dt}\langle X,\nu\rangle - F^{ij}\langle X,\nu\rangle_{;ij} = -F^{ij}h_{ij} - F + F^{ij}h_i^k h_{kj}\langle X,\,\nu\rangle. \tag{3.12}$$

Proof. We obtain
$$\begin{aligned}\frac{d}{dt}\langle X,\,\nu\rangle - F^{ij}\langle X,\,\nu\rangle_{;ij} =& X^\alpha\delta_{\alpha\beta}\left(\frac{d}{dt}\nu^\beta - F^{ij}\nu^\alpha_{;ij}\right)\\ &+\left(\frac{d}{dt}X^\alpha - F^{ij}X^\alpha_{;\,ij}\right)\delta_{\alpha\beta}\nu^\beta\\ &- 2F^{ij}X^\alpha_{;\,i}\delta_{\alpha\beta}\nu^\beta_{;\,j}\\ =& F^{ij}h_i^k h_{kj}\langle X,\,\nu\rangle + \left(F^{ij}h_{ij} - F\right)\langle\nu,\,\nu\rangle\\ &- 2F^{ij}X^\alpha_{;\,i}\delta_{\alpha\beta}h_j^k X^\beta_{;\,k}\end{aligned}$$
by (1.3), (3.8), and (3.11)

$$= F^{ij}h_i^k h_{kj}\langle X,\,\nu\rangle - F^{ij}h_{ij} - F.$$
□

Lemma 3.10. *Let* $\eta_\alpha = (-e_{n+1})_\alpha = (0,\ldots,0,-1)$. *Then* $\tilde{v} := \langle\eta,\nu\rangle \equiv \eta_\alpha\nu^\alpha$ *fulfills*

$$\frac{d}{dt}\tilde{v} - F^{ij}\tilde{v}_{;ij} = F^{ij}h_i^k h_{kj}\tilde{v} \tag{3.13}$$

and $v := \tilde{v}^{-1}$ *fulfills*

$$\frac{d}{dt}v - F^{ij}v_{;ij} = -\,vF^{ij}h_i^k h_{kj} - 2\frac{1}{v}F^{ij}v_iv_j. \tag{3.14}$$

Proof. The evolution equation for $\tilde{v}$ is a direct consequence of (3.11). For the proof of the evolution equation of v observe that

$$v_i = -\,\tilde{v}^{-2}\tilde{v}_i = -v^2\tilde{v}_i$$

and

$$v_{;ij} = -\,\tilde{v}^{-2}\tilde{v}_{;ij} + 2\tilde{v}^{-3}\tilde{v}_i\tilde{v}_j = -v^2\tilde{v}_{;ij} + 2v^{-1}v_iv_j.$$

□

4. Graphical Solutions to Mean Curvature Flow

For mean curvature flow of entire graphs, K. Ecker and G. Huisken proved the following existence theorem [18, Theorem 5.1]

Theorem 4.1. *Let* $u_0 : \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz continuous. Then there exists a function* $u \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$ *solving*

$$\begin{cases} \dot{u} = \sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } \mathbb{R}^n \times (0,\infty),\\ u(\cdot,t) \to u_0 & \text{as } t \searrow 0 \text{ in } C^0_{loc}(\mathbb{R}^n). \end{cases}$$

Strategy of proof.

(i) Approximate u_0 by smooth functions. Hence we will assume in the following that u_0 is smooth.

(ii) Consider $R > 0$. Eventually, we will let $R \to \infty$ in order to obtain a solution. Consider graphical mean curvature flow with initial condition u_R,

$$\begin{cases} \dot{u} = \sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } B_{3R}(0) \times (0,\infty),\\ u(\cdot,0) = u_R & \text{in } B_{3R}(0),\\ u(\cdot,t) = 0 & \text{on } \partial B_{3R}(0), \end{cases}$$

where u_R is smooth, $u_R = u_0$ in $B_R(0)$, and $u_R = 0$ in $B_{3R}(0) \setminus B_{2R}(0)$. Solutions to that initial value problem are known to exist for all times $t \ge 0$ and they are smooth. We call the solution to that initial value problem u^R.

(iii) In order to be able to let $R \to \infty$ and to obtain a subsequence that converges to a solution, we need interior estimates. For the first spatial derivatives, we will prove such an estimate in Theorem 4.2. For higher order spatial derivatives, there are a priori estimates for the second fundamental form A of the form

$$\sup_{B_{\frac12 R}(x_0)} |\nabla^m A|^2(\cdot,t) \le c\left(m, n, \sup_{B_R(x_0)\times[0,t]} |Du|\right) \cdot \left(\frac{1}{R^2} + \frac{1}{t}\right)^{m+1},$$

if a solution to mean curvature flow can be written as a graph over a set $B_R(x_0)$, see [18, Corollary 3.5] for details.

(iv) Apply Arzelà-Ascoli and obtain a subsequence that converges to a solution u as desired.

□

The following result is Theorem 2.3 in [18], obtained by K. Ecker and G. Huisken,

Theorem 4.2. *Let $u : B_R(0) \times [0,T] \to \mathbb{R}$ be a smooth solution to graphical mean curvature flow. Then*

$$\sqrt{1+|Du|^2(0,t)} \le c(n) \sup_{B_R(0)} \sqrt{1+|Du|^2(\cdot,0)} \cdot \exp\left(c(n)\,R^{-2}\left(\operatorname*{osc}_{B_R(0)\times[0,T]} u\right)^2\right).$$

Proof. Remember that $\operatorname{osc} u := \sup u - \inf u$. Assume first that $u \ge 0$ in $B_R(0) \times [0,T]$ by considering $u - \inf\limits_{B_R(0)\times[0,T]} u$ instead of u. (In the following definition of φ, $u > 0$ corresponds to $\langle X, \eta\rangle < 0$ for $\eta = (0,\ldots,0,-1)$. We will see, however, that $\frac{d}{dt}\langle X,\eta\rangle - \Delta\langle X,\eta\rangle = 0$, whereas $\dot{u} = vH$. The difference comes from the fact, that $\dot{u}$ describes the speed in vertical direction, whereas mean curvature flow prescribes the normal velocity.) Scaling the solution according to $\tilde{u}(x,t) := \frac{1}{R}u\left(Rx, R^2t\right)$, we obtain another solution to mean curvature flow, defined on $B_1(0)\times\left[0, \frac{T}{R^2}\right]$. Proving the theorem for $\tilde{u}$ is equivalent to proving the theorem for u. Let us therefore assume that $R = 1$.

Consider the function $v\psi$ with $\psi := -1 + \exp(\lambda\varphi)$ and

$$\varphi := \left(\frac{1}{2\beta}\langle X,\eta\rangle + 1 - \left(|X|^2 - \langle X,\eta\rangle^2\right)\right)_+,$$

where $\lambda, \beta > 0$ are constants to be chosen and $\eta = (0,\ldots,0,-1)$. Assume that for the first time t_0, $v\psi$ reaches a new positive maximum. Then ψ is positive and hence smooth near the maximum and we get there

$$(v\psi)_{;i} = 0, \quad \frac{d}{dt}(v\psi) - \Delta(v\psi) \equiv \frac{d}{dt}(v\psi) - g^{ij}(v\psi)_{;ij} \ge 0.$$

The following calculations are also valid in this new maximum. Using the evolution equation for v, (3.14), $F^{ij} = H^{ij} = g^{ij}$, and the extremal condition, we obtain

$$\begin{aligned} 0 \le{}& \left(\frac{d}{dt}v - \Delta v\right)\psi + v\left(\frac{d}{dt}\psi - \Delta\psi\right) - 2g^{ij}v_{;i}\psi_{;j} \\ ={}& -vg^{ij}h_i^k h_{kj}\psi - 2\frac{\psi}{v}g^{ij}v_iv_j + v\left(\frac{d}{dt}\psi - \Delta\psi\right) - 2g^{ij}v_{;i}\psi_{;j} \\ \equiv{}& -v|A|^2\psi - 2\frac{\psi}{v}|\nabla v|^2 + v\left(\frac{d}{dt}\psi - \Delta\psi\right) - 2\langle\nabla v, \nabla\psi\rangle \\ ={}& -v|A|^2\psi - 2\frac{\psi}{v}|\nabla v|^2 + v\left(\frac{d}{dt}\psi - \Delta\psi\right) + 2\left\langle\nabla v, \frac{\psi\nabla v}{v}\right\rangle \\ ={}& -v|A|^2\psi + v\left(\frac{d}{dt}\psi - \Delta\psi\right). \end{aligned}$$

As $v > 0$, $\psi \ge 0$, we get

$$\frac{d}{dt}\psi - \Delta\psi \ge 0.$$

According to the definition of ψ,

$$\begin{aligned} \psi_{;i} &= e^{\lambda\varphi}\lambda\varphi_{;i}, \\ \psi_{;ij} &= e^{\lambda\varphi}\left(\lambda\varphi_{;ij} + \lambda^2\varphi_{;i}\varphi_{;j}\right) \end{aligned}$$

and hence

$$\frac{d}{dt}\varphi - \Delta\varphi \ge \lambda|\nabla\varphi|^2. \tag{4.1}$$

Due to the definition of φ, we see that

$$\begin{aligned}
\varphi_{;i} &= \frac{1}{2\beta}X_i^\alpha\eta_\alpha - \left(|X|^2 - \langle X,\eta\rangle^2\right)_i,\\
|\nabla\varphi|^2 &= \frac{1}{4\beta^2}\eta_\alpha X_i^\alpha g^{ij}X_j^\beta\eta_\beta + \left|\nabla\left(|X|^2 - \langle X,\eta\rangle^2\right)\right|^2\\
&\quad - \frac{1}{\beta}\eta_\alpha X_i^\alpha g^{ij}\left(|X|^2 - \langle X,\eta\rangle^2\right)_j\\
&\ge \frac{1}{4\beta^2}\eta_\alpha X_i^\alpha g^{ij}X_j^\beta\eta_\beta - \frac{1}{\beta}\eta_\alpha X_i^\alpha g^{ij}\left(|X|^2 - \langle X,\eta\rangle^2\right)_j,\\
\frac{d}{dt}\varphi - \Delta\varphi &= \frac{1}{2\beta}\left(\frac{d}{dt}\langle X,\eta\rangle - \Delta\langle X,\eta\rangle\right)\\
&\quad - \left\{\frac{d}{dt}\left(|X|^2 - \langle X,\eta\rangle^2\right) - \Delta\left(|X|^2 - \langle X,\eta\rangle^2\right)\right\}\\
&= \frac{1}{2\beta}\left\langle\frac{d}{dt}X - \Delta X,\eta\right\rangle - 2\left\langle X,\frac{d}{dt}X - \Delta X\right\rangle\\
&\quad + 2\langle X,\eta\rangle\left\langle\frac{d}{dt}X - \Delta X,\eta\right\rangle + 2g^{ij}X_i^\alpha X_j^\beta\bar g_{\alpha\beta} - 2g^{ij}X_i^\alpha X_j^\beta\eta_\alpha\eta_\beta\\
&\le 2g^{ij}X_i^\alpha X_j^\beta\bar g_{\alpha\beta}\\
&= 2g^{ij}g_{ij}\\
&= 2n,
\end{aligned}$$

where we have used that

$$\frac{d}{dt}X - \Delta X = -H\nu - g^{ij}X_{;ij} = -H\nu + g^{ij}h_{ij}\nu = 0.$$

Using (4.1) and the estimate for $|\nabla\varphi|^2$, we deduce that

$$\lambda\left(\frac{1}{4\beta^2}\eta_\alpha X_i^\alpha g^{ij}X_j^\beta\eta_\beta - \frac{1}{\beta}\eta_\alpha X_i^\alpha g^{ij}\left(|X|^2 - \langle X,\eta\rangle^2\right)_j\right) \le 2n. \tag{4.2}$$

Note that $\bar g^{\alpha\beta} \equiv \delta^{\alpha\beta} = X_i^\alpha g^{ij}X_j^\beta + \nu^\alpha\nu^\beta$. This can be verified by testing the equation with the unit normal ν and tangent vectors X_i. Hence

$$\begin{aligned}
\eta_\alpha X_i^\alpha g^{ij}X_j^\beta\eta_\beta &= \eta_\alpha\bar g^{\alpha\beta}\eta_\beta - \eta_\alpha\nu^\alpha\nu^\beta\eta_\beta\\
&= 1 - \langle\eta,\nu\rangle^2\\
&= 1 - v^{-2},\\
\eta_\alpha X_i^\alpha g^{ij}\left(|X|^2 - \langle X,\eta\rangle^2\right)_j &= \eta_\alpha X_i^\alpha g^{ij}\cdot X_j^\gamma\left(2X^\beta\bar g_{\beta\gamma} - 2\langle X,\eta\rangle\eta_\gamma\right)\\
&= 2\eta_\alpha\left(\bar g^{\alpha\gamma} - \nu^\alpha\nu^\gamma\right)\left(X^\beta\bar g_{\beta\gamma} - \langle X,\eta\rangle\eta_\gamma\right)\\
&= 2\langle\eta,X\rangle - 2\langle\eta,\nu\rangle\langle\nu,X\rangle\\
&\quad - 2|\eta|^2\langle X,\eta\rangle + 2\langle X,\eta\rangle\langle\eta,\nu\rangle^2
\end{aligned}$$

$$= -2\langle \eta, \nu\rangle\langle \nu, X\rangle + 2\langle X, \eta\rangle\langle \eta, \nu\rangle^2.$$

In coordinates, we have $\eta = (0, -1)$, $\nu = v^{-1}(Du, -1)$, and $X = (x, u)$. According to the definition of φ and in view of $\langle X, \eta\rangle < 0$, $v\psi \equiv 0$ if $|x|^2 + u^2 \geq 1$. Hence we may assume that $|x| < 1$ and $|u| < 1$. We compute

$$\begin{aligned} -2\langle \eta, \nu\rangle\langle \nu, X\rangle + 2\langle X, \eta\rangle\langle \eta, \nu\rangle^2 =& -2v^{-1}v^{-1}(\langle Du, x\rangle - u) - 2uv^{-2} \\ \leq& \, 2v^{-2}|Du| \cdot 1 \\ \leq& \, 2v^{-1}. \end{aligned}$$

Combining these estimates with (4.2), we get

$$\lambda\left(\frac{1}{4\beta^2}(1 - v^{-2}) - \frac{2}{\beta}v^{-1}\right) \leq 2n.$$

Now we define $\lambda := 64n\beta^2$ and obtain (by a direct calculation) $7v^2 - 64\beta v - 8 \leq 0$ and $v \leq 4 + 16\beta$. Note that $\varphi \leq 1$. We denote $\operatorname{graph} u(\cdot, t)|_{B_1(0)} \cap \{\psi > 0\}$ by $\tilde{M}_t$. Hence

$$\max_{\tilde{M}_{t_0}} v\psi \leq (4 + 16\beta) \sup_{\tilde{M}_{t_0}} \psi \leq (4 + 16\beta)e^{\lambda} = (4 + 16\beta)e^{64n\beta^2}$$

and thus, as we have considered an arbitrary new increasing maximum of $v\psi$,

$$v\psi \leq \sup_{\tilde{M}_0} v\psi + (4 + 16\beta)e^{64n\beta^2} \leq e^{64n\beta^2}\left(4 + 16\beta + \sup_{B_1(0)} \sqrt{1 + |Du|^2(\cdot, 0)}\right)$$

everywhere on $\tilde{M}_t$, $0 \leq t < T$. We set $\beta := 1 + \sup\limits_{t\in[0,T]} u(0, t)$. Therefore, we get for $t \in [0, T]$ and $x = 0$

$$\begin{aligned} v\psi =& \sqrt{1 + |Du|^2(0, t)}\left(-1 + e^{64n\beta^2\left(\frac{-u(0,t)}{2+2\sup\limits_{t\in[0,T]} u(0,t)} + 1\right)_+}\right) \\ \geq& \sqrt{1 + |Du|^2(0, t)}\left(-1 + e^{64n\left(-\frac{1}{2}+1\right)}\right) \\ \geq& \sqrt{1 + |Du|^2(0, t)}. \end{aligned}$$

We conclude that

$$\sqrt{1 + |Du|^2(0, t)} \leq c(n) \sup_{B_1(0)\times\{0\}} \sqrt{1 + |Du|^2} \cdot \exp\left(c(n) \sup_{t\in[0,T]} u(0, t)^2\right).$$

The theorem follows. $\square$

Theorem 4.1 has been extended to continuous initial data by J. Clutterbuck [13] and T. Colding and W. Minicozzi [15].

If u is initially close to a cone in an appropriate sense, graphical mean curvature flow converges, as $t \to \infty$, after appropriate rescaling, to a self-similarly expanding solution "coming out of a cone", see the papers by K. Ecker and G. Huisken [18] and N. Stavrou [44].

Stability of translating solutions to graphical mean curvature flow without rescaling is considered in [14].

5. Convex Hypersurfaces

G. Huisken obtained the following theorem [27] for $n \geq 2$. The corresponding result for curves by M. Gage, R. Hamilton, and M. Grayson is even better, see [21, 24]. It is only required that $M \subset \mathbb{R}^2$ is a closed embedded curve.

Theorem 5.1. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth closed convex hypersurface. Then there exists a smooth family M_t of hypersurfaces solving*

$$\begin{cases} \frac{d}{dt}X = -H\nu & \text{for } 0 \leq t < T, \\ M_0 = M \end{cases}$$

for some $T > 0$.

As $t \nearrow T$,

- *$M_t \to Q$ in Hausdorff distance for some $Q \in \mathbb{R}^{n+1}$ (convergence to a point),*
- *$(M_t - Q) \cdot (2n(T-t))^{-1/2} \to \mathbb{S}^n$ smoothly (convergence to a "round point").*

The key step in the proof of Theorem 5.1 (in the case $n \geq 2$) is the following

Theorem 5.2. *Let $M_t \subset \mathbb{R}^{n+1}$ be a family of convex closed hypersurfaces flowing according to mean curvature flow. Then there exists some $\delta > 0$ such that*

$$\max_{M_t} \frac{n|A|^2 - H^2}{H^{2-\delta}}$$

is bounded above.

The proof involves complicated integral estimates.

Exercise 5.3. Prove Theorem 5.2 for $\delta = 0$.

Remark 5.4. For simplicity, we will illustrate the significance of the quantity considered in Theorem 5.2 only in the case $n = 2$. These considerations extend to higher dimensions.

As

$$\begin{aligned} 2|A|^2 - H^2 &= 2(\lambda_1^2 + \lambda_2^2) - (\lambda_1 + \lambda_2)^2 \\ &= 2\lambda_1^2 + 2\lambda_2^2 - \lambda_1^2 - 2\lambda_1\lambda_2 - \lambda_2^2 \\ &= \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 \\ &= (\lambda_1 - \lambda_2)^2, \end{aligned}$$

it measures the difference from being umbilic ($\lambda_1 = \lambda_2$) and vanishes precisely if M_t is a sphere. Recall from differential geometry that, according to Codazzi, $\lambda_1 = \lambda_2$ everywhere implies that M_t is locally part of a sphere or hyperplane.

Assume that $\min\limits_{M_t} H \to \infty$ as $t \nearrow T$. Assume also that $\lambda_1 \leq \lambda_2$ and that the surfaces stay strictly convex, i. e. $\min\limits_{M_t} \lambda_1 > 0$. Then Theorem 5.2 implies for any ε there exists t_ε, such that for $t_\varepsilon \leq t < T$

$$\varepsilon \geq \frac{n|A|^2 - H^2}{H^2} = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2} \geq \frac{(\lambda_1 - \lambda_2)^2}{4\lambda_2^2} = \frac{1}{4}\left(\frac{\lambda_1}{\lambda_2} - 1\right)^2.$$

Hence $\frac{\lambda_1}{\lambda_2} \approx 1$ and thus this implies that M_t is, in terms of the principal curvatures λ_i, close to a sphere.

There are many results showing that convex hypersurfaces converge to round points under certain flow equations, see e. g. [1, 2, 9, 20, 21, 23, 31, 39, 40, 48].

Let us consider normal velocities of homogeneity bigger than one. In this case, the calculations, that lead to a theorem corresponding to Theorem 5.2 for mean curvature flow, are much simpler and rely only on the maximum principle.

Theorem 5.5. *[*[2, Proposition 3]*] Let M_t be a smooth family of closed strictly convex solutions to Gauß curvature flow $\frac{d}{dt}X = -K\nu$. Then*

$$t \mapsto \max_{M_t} (\lambda_1 - \lambda_2)^2$$

is non-increasing.

Proof. Recall that $H^2 - 4K = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 =: w$. For Gauß curvature flow, we have, according to Appendix B,

$$\begin{aligned} F^{ij} = &K^{ij} = \frac{\partial}{\partial h_{ij}} \frac{\det h_{kl}}{\det g_{kl}} = \frac{\det h_{kl}}{\det g_{kl}} \tilde{h}^{ij} = K\tilde{h}^{ij}, \\ F^{ij,kl} = &K\tilde{h}^{ij}\tilde{h}^{kl} - K\tilde{h}^{ik}h^{lj}, \end{aligned}$$

where $\tilde{h}^{ij}$ is the inverse of h_{ij}. Recall the evolution equations (3.2), (3.6), and (3.7) which become for Gauß curvature flow

$$\begin{aligned} \frac{d}{dt} g_{ij} = &-2Kh_{ij}, \\ \frac{d}{dt}K - K\tilde{h}^{kl}K_{kl} = &KK\tilde{h}^{ij}h_i^k h_{kj} \\ = &K^2 H, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}h_{ij} - K\tilde{h}^{kl}h_{ij;kl} = &K\tilde{h}^{kl}h_k^a h_{al}h_{ij} - K\tilde{h}^{kl}h_{kl}h_i^a h_{aj} - Kh_i^k h_{kj} \\ &+ K\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j} \\ = &KHh_{ij} - (n+1)Kh_i^a h_{aj} + K\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j}, \end{aligned}$$

where $n = 2$. We have

$$\begin{aligned} \frac{d}{dt}H - K\tilde{h}^{ij}H_{;ij} = &-h_{ij}g^{ik}g^{jl}\frac{d}{dt}g_{kl} + g^{ij}\left(\frac{d}{dt}h_{ij} - K\tilde{h}^{kl}h_{ij;kl}\right) \\ = &2K|A|^2 + KH^2 - 3K|A|^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j} \\ = &K\left(H^2 - |A|^2\right) + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j} \\ = &2K^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j}, \end{aligned}$$

hence

$$\begin{aligned}\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} = & \, 2H\left(\frac{d}{dt}H - K\tilde{h}^{ij}H_{;ij}\right) - 2K\tilde{h}^{ij}H_iH_j \\ & - 4\left(\frac{d}{dt}K - K\tilde{h}^{ij}K_{;ij}\right) \\ = & \, 2H\left(2K^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}\right) \\ & - 2K\tilde{h}^{ij}H_iH_j - 4K^2H \\ = & \, 2HKg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j} - 2K\tilde{h}^{ij}H_iH_j.\end{aligned}$$

In a coordinate system, such that $g_{ij} = \delta_{ij}$ and $h_{ij} = \operatorname{diag}(\lambda_1, \lambda_2)$, we obtain

$$\begin{aligned}\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} = & \, 2KH\sum_{i,j,k=1}^{2}\frac{1}{\lambda_i\lambda_j}h_{ii;k}h_{jj;k} - 2KH\sum_{i,j,k=1}^{2}\frac{1}{\lambda_i\lambda_j}h^2_{ij;k} \\ & - 2K\sum_{i,j,k=1}^{2}\frac{1}{\lambda_k}h_{ii;k}h_{jj;k} \\ = & \, 2KH\sum_{\substack{i,j,k=1\\ i\neq j}}^{2}\frac{1}{\lambda_i\lambda_j}h_{ii;k}h_{jj;k} - 2KH\sum_{\substack{i,j,k=1\\ i\neq j}}^{2}\frac{1}{\lambda_i\lambda_j}h^2_{ij;k} - 2K\sum_{i,j,k=1}^{2}\frac{1}{\lambda_k}h_{ii;k}h_{jj;k} \\ = & \, \frac{4KH}{\lambda_1\lambda_2}\left(h_{11;1}h_{22;1} - h^2_{12;1} + h_{11;2}h_{22;2} - h^2_{12;2}\right) \\ & - \frac{2K}{\lambda_1}(h_{11;1} + h_{22;1})^2 - \frac{2K}{\lambda_2}(h_{11;2} + h_{22;2})^2.\end{aligned}$$

From now on, we consider a positive spatial maximum of $H^2 - 4K$. There, we get $2Hg^{ij}h_{ij;k} - 4K\tilde{h}^{ij}h_{ij;k} = 0$ for $k = 1, 2$. In a coordinate system as above, this (divided by 2) becomes

$$\begin{aligned}0 = & \, Hh_{11;k} + Hh_{22;k} - 2\frac{K}{\lambda_1}h_{11;k} - 2\frac{K}{\lambda_2}h_{22;k} \\ = & \, (\lambda_1 + \lambda_2 - 2\lambda_2)h_{11;k} + (\lambda_1 + \lambda_2 - 2\lambda_1)h_{22;k} \\ = & \, (\lambda_1 - \lambda_2)(h_{11;k} - h_{22;k}).\end{aligned}$$

This enables us to replace $h_{11;2}$ in the evolution equation in a positive critical point by $h_{22;2}$. Using also the Codazzi equations, we can rewrite the evolution equation in a positive critical point as

$$\begin{aligned}\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} = & \, 4(\lambda_1 + \lambda_2)\left(h^2_{11;1} - h^2_{22;2} + h^2_{22;2} - h^2_{11;1}\right) \\ & - \frac{2K}{\lambda_1}(h_{11;1} + h_{22;1})^2 - \frac{2K}{\lambda_2}(h_{11;2} + h_{22;2})^2 \\ \le & \, 0.\end{aligned}$$

Hence, by the parabolic maximum principle, Theorem A.1, the claim follows. ☐

A consequence of Theorem 5.5 is the following result, see [2, Theorem 1].

Theorem 5.6. *Let $M \subset \mathbb{R}^3$ be a smooth closed strictly convex surface. Then there exists a smooth family of closed strictly convex hypersurfaces solving Gauß curvature flow $\frac{d}{dt}X = -K\nu$ for $0 \le t < T$. As $t \nearrow T$, M_t converges to a round point.*

Sketch of proof. The main steps are

(i) The convergence to a point is due to K. Tso [47]. There, the problem is rewritten in terms of the support function and considered in all dimensions. It is shown that a positive lower bound on the Gauß curvature es preserved during the evolution. This ensures that the surfaces stay convex. The evolution equation of $\frac{K}{\langle X,\nu\rangle - \frac{1}{2}R}$ is used to bound the principal curvatures as long as the surface encloses $B_R(0)$. Thus a positive lower bound on the principal curvatures follows. Parabolic Krylov-Safonov estimates imply bounds on higher derivatives.

(ii) Theorem 5.5,

(iii) Show that M_t is between spheres of radius $r_+(t)$ and $r_-(t)$ and center $q(t)$ with $\frac{r_+(t)}{r_-(t)} \to 1$ as $t \nearrow T$.

(iv) Show that the quotient $\frac{K(p,t)}{K_{r(t)}}$ converges to 1 as $t \nearrow T$. Here $r(t) = (3(T-t))^{1/3}$ is the radius of a sphere flowing according to Gauß curvature flow that becomes singular at $t = T$ and $K_{r(t)} = (3(T-t))^{-2/3}$ its Gauß curvature. This involves a Harnack inequality for the normal velocity.

(v) Show that $\frac{\lambda_i}{(3(T-t))^{-1/3}} \to 1$ as $t \nearrow T$.

(vi) Obtain uniform a priori estimates for a rescaled version of the flow and hence smooth convergence to a round sphere.

□

We see directly from the parabolic maximum principle for tensors that a positive lower bound on the principal curvatures is preserved for surfaces moving with normal velocity $|A|^2$.

Lemma 5.7. *For a smooth closed strictly convex surface M in $\mathbb{R}^3$, flowing according to $\frac{d}{dt}X = -|A|^2\nu$, the minimum of the principal curvatures is non-decreasing.*

Proof. We have $F = |A|^2 = h_{ij}g^{jk}h_{kl}g^{li}$, $F^{ij} = 2g^{ia}h_{ab}g^{bj}$, and $F^{ij,kl} = 2g^{ik}g^{jl}$. Consider $M_{ij} = h_{ij} - \varepsilon g_{ij}$ with $\varepsilon > 0$ so small that M_{ij} is positive semi-definite for some time t_0. We wish to show that M_{ij} is positive semi-definite for $t > t_0$. Using (3.7), we obtain

$$\frac{d}{dt}h_{ij} - F^{kl}h_{ij;\,kl} = 2\,\mathrm{tr}\,A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2g^{kr}g^{ls}h_{kl;\,i}h_{rs;\,j}.$$

In the evolution equation for M_{ij}, we drop the positive definite terms involving derivatives of the second fundamental form

$$\frac{d}{dt}M_{ij} - F^{kl}M_{ij;\,kl} \ge 2\,\mathrm{tr}\,A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}.$$

Let ξ be a zero eigenvalue of M_{ij} with $|\xi|=1$, $M_{ij}\xi^j=h_{ij}\xi^j-\varepsilon g_{ij}\xi^j=0$. So we obtain in a point with $M_{ij}\ge 0$

$$\begin{aligned}\left(2\operatorname{tr}A^3h_{ij}-3|A|^2h_i^kh_{kj}+2\varepsilon|A|^2h_{ij}\right)\xi^i\xi^j=&2\varepsilon\operatorname{tr}A^3-3\varepsilon^2|A|^2+2\varepsilon^2|A|^2\\=&2\varepsilon\operatorname{tr}A^3-\varepsilon^2|A|^2\\\ge&2\varepsilon^2|A|^2-\varepsilon^2|A|^2>0\end{aligned}$$

and the maximum principle for tensors, Theorem A.2, which extends to the case $\frac{d}{dt}M_{ij}\ge\ldots$, gives the result. □

Exercise 5.8. Show that under mean curvature flow of closed hypersurfaces, the following inequalities are preserved during the flow.

(i) $0\le H$, $0<H$,
(ii) $h_{ij}\ge 0$,
(iii) $\varepsilon Hg_{ij}\le h_{ij}\le\beta Hg_{ij}$ for $0<\varepsilon\le\frac{1}{n}<\beta<1$.

Such estimates exist also for other normal velocities.

Theorem 5.9 ([39])**.** *Let M_t be a family of closed strictly convex hypersurfaces evolving according to $\frac{d}{dt}X=-|A|^2\nu$. Then*

$$t\mapsto\max_{M_t}\frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$$

is non-increasing.

Exercise 5.10.

(i) Prove Theorem 5.9.

Hint: In a positive critical point of $w:=\frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$, for $F=|A|^2$, the evolution equation of w is given by

$$\begin{aligned}\frac{d}{dt}w-F^{ij}w_{;ij}=&-4(\lambda_1-\lambda_2)^2\lambda_1\lambda_2\\&-2\frac{5\lambda_1^8-4\lambda_1^7\lambda_2+46\lambda_1^6\lambda_2^2+48\lambda_1^5\lambda_2^3+72\lambda_1^4\lambda_2^4}{\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)^2\lambda_1^4}h_{11;1}^2\\&-2\frac{44\lambda_1^3\lambda_2^5+34\lambda_1^2\lambda_2^6+8\lambda_1\lambda_2^7+3\lambda_2^8}{\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)^2\lambda_1^4}h_{11;1}^2\\&-2\frac{5\lambda_2^8-4\lambda_2^7\lambda_1+46\lambda_2^6\lambda_1^2+48\lambda_2^5\lambda_1^3+72\lambda_2^4\lambda_1^4}{\left(\lambda_2^2+\lambda_2\lambda_1+\lambda_1^2\right)^2\lambda_2^4}h_{22;2}^2\\&-2\frac{44\lambda_2^3\lambda_1^5+34\lambda_2^2\lambda_1^6+8\lambda_2\lambda_1^7+3\lambda_1^8}{\left(\lambda_2^2+\lambda_2\lambda_1+\lambda_1^2\right)^2\lambda_2^4}h_{22;2}^2.\end{aligned}$$

(This is a longer calculation.)

(ii) Show that the only closed strictly convex surfaces contracting self-similarly (by homotheties) under $\frac{d}{dt}X=-|A|^2\nu$, are round spheres. A surface M_t is said to evolve by homotheties, if for every t_1,t_2, there exists $\lambda\in\mathbb{R}$ such that $M_{t_1}=\lambda M_{t_2}$.

(iii) Show that for closed strictly convex initial data M, there exists some $c>0$ such that $\frac{1}{c}\le\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}\le c$ for surfaces evolving according to $\frac{d}{dt}X=-|A|^2\nu$ for all $0\le t<T$, where T is, as usual, the maximal existence time.

Similar results also exist for expanding surfaces

Theorem 5.11 ([40]). *Let M_t be a family of closed strictly convex hypersurfaces evolving according to $\frac{d}{dt}X = \frac{1}{K}\nu$. Then*

$$t \mapsto \max_{M_t} \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2}$$

is non-increasing.

Exercise 5.12. Prove Theorem 5.11 and deduce consequences similar to those in Exercise 5.10.

Hint: In a critical point of $w := \frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2\lambda_2^2}$, the evolution equation of w reads

$$\frac{d}{dt}w - F^{ij}w_{;ij} = -2\frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1^3\lambda_2^3} - \frac{8}{\lambda_1^6\lambda_2}h^2_{11;\,1} - \frac{8}{\lambda_1\lambda_2^6}h^2_{22;\,2}.$$

5.1. **Isoperimetric Inequalities.** In a situation, where we know, that mean curvature flow exists until the enclosed volume shrinks to zero, it can be used to prove isoperimetric inequalities. Flow equations and isoperimetric inequalities are studied by G. Huisken, F. Schulze [41], and P. Topping [45]. We want to describe such an approach in a model situation.

Consider a family M_t^2 of smooth closed surfaces that moves by mean curvature flow until the volume of the enclosed area Ω_t shrinks to zero. In this case, the isoperimetric inequality reads

$$\frac{1}{3\sigma}\left(\mathcal{H}^2(M_t)\right)^{3/2} - \mathcal{H}^3(\Omega_t) \equiv \frac{1}{3\sigma}|M_t|^{3/2} - |\Omega_t| \ge 0$$

for $\sigma = \sqrt{4\pi}$. (Recall that $|\partial B_1^2| = 4\pi$ and $|B_1^2| = \frac{4\pi}{3}$.) We want to prove this inequality using mean curvature flow for surfaces M_t which are topologically spheres. Note first that by Hölder's inequality

$$\int\limits_{M_t} H \le \left(\int\limits_{M_t} 1\right)^{1/2} \left(\int\limits_{M_t} H^2\right)^{1/2}.$$

Secondly, by Gauß-Bonnet, as we are on a topological sphere,

$$\int\limits_{M_t} H^2 = \int\limits_{M_t} (\lambda_1-\lambda_2)^2 + \int\limits_{M_t} 4K \ge 4\int\limits_{M_t} K = 4\mathcal{H}^2(\partial B_1(0)) \equiv 4|\partial B_1(0)| = 16\pi.$$

Hence, under the evolution by mean curvature flow, we get according to (3.3) the following estimate for the isoperimetric difference (we may assume that $\int_{M_t} H > 0$ for otherwise the inequality derived in the following follows already from the first line)

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{3\sigma}|M_t|^{3/2} - |\Omega_t|\right) &= \frac{1}{2\sigma}|M_t|^{1/2}\int\limits_{M_t} -H^2\,d\mu - \int\limits_{M_t} -H\,d\mu \\ &\le -\frac{1}{2\sigma}\left(\int\limits_{M_t} H^2\right)^{1/2}\int\limits_{M_t} H + \int\limits_{M_t} H\end{aligned}$$

$$= \left(1 - \frac{1}{2\sigma}\left(\int\limits_{M_t} H^2\right)^{1/2}\right)\int\limits_{M_t} H$$
$$\le \left(1 - \frac{1}{2\sigma}\sqrt{16\pi}\right)\int H$$
$$= 0.$$

Setting $f(t) := \frac{1}{3\sigma}|M_t|^{3/2} - |\Omega_t|$, we have $f(T) \ge 0$ (considered as a limit as $t \nearrow T$) as we have assumed that $|\Omega_T| = 0$ (in the sense of a limit). Integrating backwards in time yields $f(t) \ge 0$ for $t < T$, which is the isoperimetric inequality claimed above.

6. Mean Curvature Flow with Surgery

We consider mean curvature flow. Even for a smooth initial surface which is topologically a sphere, singularities will occur before the surface can shrink to a point.

The following example can be found in greater generality in [17].

Example 6.1. In order to see this, we consider a family of hypersurfaces in $\mathbb{R}^3$ given by

$$M_t = \left\{(x, y, z) \in \mathbb{R}^3 : \tfrac{1}{2}z^2 = x^2 + y^2 - 1 + t\right\}$$

for $0 \le t < 1$. For $t \nearrow 1$, these hypersurfaces converge to a cone with singularity at the origin. We want to see that this hypersurface is a barrier for mean curvature flow. Locally, we can write it near $y = 0$ as a graph $x = \sqrt{\frac{1}{2}z^2 - y^2 + 1 - t}$. Due to the symmetry, it suffices to understand the evolution near $y = 0$. Observe that the function $u(x^1, x^2) := \sqrt{\frac{1}{2}(x^1)^2 - (x^2)^2 + 1 - t}$ fulfills

$$\dot{u} \ge \sqrt{1 + |Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right)$$

for $0 \le t < 1$ and $x^2 \approx 0$. This is a direct calculation (exercise). That is, the surface M_t shrinks slower in the direction of the axis $z = 0$ than a surface flowing by mean curvature flow. Hence we can apply a comparison principle principle (proved exactly as the avoidance principle, Theorem 2.16, using the inequality in the maximum principle) to see that any compact surface flowing according to mean curvature flow which is initially in the set $\{\frac{1}{2}z^2 \ge x^2 + y^2 - 1 + 0\}$, will be contained in the set $\{\frac{1}{2}z^2 \ge x^2 + y^2 - 1 + t\}$ for all $0 \le t < 1$, if it evolves smoothly. Note that the surfaces M_t are not compact, but the avoidance principle still applies as it only needs that one of the surfaces is compact, as in such a case there has to be a point of first contact.

We now pick a surface N that encloses two sufficiently large balls (which evolve smoothly up to $t = 2$) on different sides of the neck of M_t. If the surface N_t evolves smoothly, it continues to enclose those balls and to be contained in the set $\left\{\frac{1}{2}z^2 \ge x^2 + y^2 - 1 + t\right\}$ for $0 \le t < 1$. Hence it has to pass through the neck. This however, is only possible, if the embedding becomes singular before $t = 1$. Therefore, singularities may occur before a surface shrinks to a point.

We have seen that in general, it is not to be expected that a smooth solution to mean curvature flow exists for smooth initial data. It is possible to study mean curvature flow in a weak setting, e. g. using a level-set flow. Another possibility is to do surgery before singularities occur. This is done by G. Huisken and C. Sinestrari in [30]. A hypersurface with principal curvatures $\lambda_1 \le \ldots \le \lambda_n$ is called 2-*convex,* if, everywhere on the hypersurface, $\lambda_1 + \lambda_2 \ge 0$. The main theorem in [30] implies

Theorem 6.2. *Let $M \subset \mathbb{R}^{n+1}$, $n \ge 3$, be a smooth closed embedded 2-convex hypersurface. Then there exists a mean curvature flow starting from M with a finite number of surgeries.*

Strategy of proof.

(i) Show that 2-convexity is preserved during mean curvature flow.
(ii) Let the surfaces evolve until the curvature becomes big somewhere.
(iii) Prove a priori estimates that allow to control the behavior of hypersurfaces near a point, where the curvature is big.
(iv) Convex parts shrink to round points and are understood. Other regions of high curvature look like a neck $\mathbb{S}^{n-1} \times [a, b]$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$, or like a degenerate neck-pinch. Necks of the form $\mathbb{S}^{n-1} \times \mathbb{S}^1$ are shown to be diffeomorphic to the model space $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Necks of the form $\mathbb{S}^{n-1} \times [a, b]$ or degenerate neck-pinches are cut out so that the curvature drops. Therefore, it has to be shown that a piece (=connected component) of M_t with a point of high curvature that is not convex or diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ has a neck-like region.
(v) Each time a surgery is performed, a certain amount of area is cut out. Hence there can be at most finitely many surgeries. In order to cut out a certain amount of area, the authors follow a neck until it opens up. This ensures the lower area bound on the differences of the areas in the surgery process.
(vi) The hard part is to control the behavior of the flow analytically. We list a few key steps to do that
   (a) In regions, where the curvature is big, the hypersurface is almost convex. For every $\delta > 0$, there exists $c(\delta, M_0) > 0$, such that
   $$\lambda_1 \ge -\delta H - c(\delta, M_0).$$
   (b) If $|\lambda_1|$ is small, then the other principal curvatures are close to each other. (Note that this corresponds to a cylinder.) It has to be shown that such a "curvature cylinder" is also a "geometric cylinder". The (pointwise) cylindrical estimate reads: For every $\eta > 0$,
   $$|\lambda_1| \le \eta H \quad \Longrightarrow \quad |\lambda_i - \lambda_j|^2 \le \eta H^2 + c(\eta, M_0, n) \quad \text{for } i, j \ge 2.$$
   (c) A pointwise gradient estimate. It states that the second fundamental form does not change much between points nearby.
   $$|\nabla A|^2 \le c(n)|A|^4 + c(M_0).$$
   (d) All this is done with few parameters that control the geometry of the hypersurfaces. Up to the choice of these parameters, the surgeries are carried out in a canonical way. It is important that the control on the geometry is uniformly maintained during the surgeries.

$\square$

Remark 6.3.

(i) The surgery is a procedure that allows to control the geometry of the hypersurfaces. Such is in general not known for other weak formulations. See, however, [28], where the Euler characteristic is controlled for a weak formulation of inverse mean curvature flow. Hence mean curvature flow with surgeries allows to reconstruct the topology of the original manifold. It turns out that all 2-convex hypersurfaces are topologically $\mathbb{S}^n$ or a finite connected sum of $\mathbb{S}^{n-1}\times\mathbb{S}^1$.
(ii) The idea to combine a geometric flow equation with a surgery procedure appears also in the work of R. Hamilton and G. Perelman on the Ricci flow.
(iii) Mean curvature flow with surgery can be considered a "canonical" way of deforming a manifold into standard pieces. Keep in mind that surgery and forming a connected sum are operations inverse to each other. Hence such procedure is a way to "geometrize" embedded 2-convex manifolds.
(iv) In Section 5, we had seen such a geometrization program for convex hypersurfaces. Without imposing curvature conditions, the possible singularities seem not to allow something like surgery. This is different in 3 dimensions if we consider abstract manifolds, i. e. manifolds which are not embedded as hypersurfaces.
(v) There is a strong similarity between extrinsic flows, i. e. flows of manifolds embedded in some other manifold like $\mathbb{R}^{n+1}$, and intrinsic flows, i. e. flows, where only the (Riemannian) metric changes in time.

7. Ricci Flow

Definition 7.1. A Riemannian metric $g(t)$ is evolving under Ricci flow with initial condition $g(0)$, a Riemannian metric, if

$$\begin{cases}\frac{\partial}{\partial t}g(t)=-2\operatorname{Ric}(g(t)),\\ g(0)=g_0.\end{cases}\tag{7.1}$$

Remark 7.2 (Existence)**.** For smooth initial data on a compact manifold, short-time existence is proved in [26] using the Nash-Moser implicit function theorem. A problem in the existence proof is that the evolution equation is degenerate parabolic, corresponding to the invariance of the Ricci flow under diffeomorphisms. In a more direct approach one considers an equivalent flow, the Ricci-DeTurck or Ricci harmonic map heat flow, which is strictly parabolic, to prove existence. This is known as the DeTurck trick [16].

Definition 7.3 (Solitons)**.** Special solutions are those that evolve only by diffeomorphisms and scaling. They are called Ricci solitons. For these solitons the metric evolves as follows

$$g(t)=\sigma(t)\varphi^*(t)g(0),$$

where $\sigma(t)>0$ and $\varphi(t)$ is a family of diffeomorphisms. This can be rewritten in terms of a constant ε corresponding to $\dot\sigma$ and a vector field X induced by $\varphi(t)$ as

$$-2\operatorname{Ric}(g_0)=\varepsilon g_0+\mathcal{L}_{X_0}g_0$$

or, equivalently, as

$$-2R_{ij}=\nabla_iX_j+\nabla_jX_i+\varepsilon(g_0)_{ij}.$$

For $\varepsilon > 0$, the soliton is called, expanding; solitons with $\varepsilon = 0$ are called steady; if $\varepsilon < 0$, the soliton is shrinking. (Note that $\mathbb{R}^n$ with its standard Euclidean metric is at the same time an expanding, a steady, and a shrinking soliton.)

If X is the gradient of a function f, the soliton equation becomes

$$R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{2} g_{ij} = 0.$$

Example 7.4 (Cigar soliton)**.** Consider $\left(\mathbb{R}^2, u(x,t)\delta\right)$, where δ denotes the standard Euclidean metric, $\lambda > 0$, and $u(x,t) = \frac{1}{\lambda|x|^2 + e^{4\lambda t}}$. Then $\left(\mathbb{R}^2, u(x,t)\delta\right)$ fulfills Ricci flow.

Exercise 7.5. Prove that in this situation, Ricci flow is equivalent to $\dot{u} = \Delta \log u$. It might be helpful to prove first that $R = -1/u \cdot \Delta \log u$. Show that the cigar soliton is a gradient soliton to Ricci flow.

8. Evolution Equations

We use the following definitions, see e. g. [26].

Definition 8.1. We define
the Christoffel symbols

$$\Gamma^h_{ij} = \tfrac{1}{2} g^{hk} (g_{jk,i} + g_{ik,j} - g_{ij,k}),$$

the Riemannian curvature tensor $R^k_{ijl} = R_{ij}{}^k{}_l$

$$R_{ij}{}^k{}_l := \Gamma^k_{jl,i} - \Gamma^k_{il,j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il},$$

and in covariant form

$$R_{ijkl} = g_{kh} R_{ij}{}^h{}_l,$$

the Ricci tensor

$$R_{ik} = g^{jl} R_{ijkl},$$

and the scalar curvature

$$R = g^{ij} R_{ij}.$$

In order to compute the evolution equation for the Riemannian curvature tensor, it is convenient to consider a normal coordinate system such that $\Gamma^k_{ij} = 0$ at the point considered. To avoid confusion about the order of differentiation, we write ∂_i and ∂_t. We obtain

$$\begin{aligned}
\partial_t \Gamma^h_{jl} &= \tfrac{1}{2} g^{hm} (\partial_j \partial_t g_{lm} + \partial_l \partial_t g_{jm} - \partial_m \partial_t g_{jl}),\\
\partial_t R_{ij}{}^h{}_l &= \partial_i \partial_t \Gamma^h_{jl} - \partial_j \partial_t \Gamma^h_{il},\\
\partial_t R_{ijkl} &= g_{hk} \partial_t R_{ij}{}^h{}_l + \partial_t g_{hk} R_{ij}{}^h{}_l,
\end{aligned}$$

and hence

$$\begin{aligned}
\partial_t R_{ijkl} = &\partial_i \partial_k R_{jl} - \partial_i \partial_l R_{jk} - \partial_j \partial_k R_{il} + \partial_j \partial_l R_{ik}\\
&- g^{pq} (R_{ijkp} R_{ql} + R_{ijpl} R_{qk}).
\end{aligned}$$

The following Lemma [26, Lemma 7.2] is a consequence of the Bianchi identities

$$0 = R_{ijkl} + R_{iklj} + R_{iljk},$$

$$0 = \partial_i R_{jklm} + \partial_j R_{kilm} + \partial_k R_{ijlm},$$

of differentiating these identities, exchanging derivatives, permuting indices and contracting.

Lemma 8.2. *We have*

$$\begin{aligned}\Delta R_{ijkl} &+ 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &= \partial_i\partial_k R_{jl} - \partial_i\partial_l R_{jk} - \partial_j\partial_k R_{il} + \partial_j\partial_l R_{ik} \\ &\quad + g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}),\end{aligned}$$

where

$$B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}.$$

Proof. Exercise (long computation). □

Based on the above considerations, we obtain the evolution equation for R_{ijkl}, see [26, Theorem 7.1],

Theorem 8.3. *The Riemannian curvature tensor satisfies the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &- g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}).\end{aligned}$$

Corollary 8.4. *The Ricci tensor and the scalar curvature fulfill the following evolution equations*

$$\frac{\partial}{\partial t}R_{ik} = \Delta R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}, \tag{8.1}$$

$$\frac{\partial}{\partial t}R = \Delta R + 2g^{ij}g^{kl}R_{ik}R_{jl}. \tag{8.2}$$

9. Spherical Space Forms

Comparable to the results in Section 5, there are results for Ricci flow that show that manifolds converge, after appropriate rescaling, smoothly to quotients of the round sphere.

The classical result in this direction is by R. Hamilton [26]

Theorem 9.1. *Let (M, g) be a compact 3-manifold of positive Ricci curvature. Then there exists a smooth solution $(M, g(t))$ to*

$$\begin{cases}\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric},\\ g(0) = g\end{cases}$$

on some finite time interval $[0, T)$. As $t \nearrow T$, after appropriate rescaling, $(M, g(t))$ converges to a manifold of constant positive curvature, i. e. of constant positive sectional curvatures.

Remark 9.2.

(i) In three dimensions,

$$R_{ijkl} = g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik} - \tfrac{1}{2}R(g_{ik}g_{jl} - g_{il}g_{jk}),$$

i. e. the Riemannian curvature tensor is determined by the Ricci tensor. In appropriate coordinates $R_{ij} = \operatorname{diag}(\lambda, \mu, \nu)$.

(ii) We list a few important estimates that are all obtained by applying the maximum principle to appropriate test functions.

(iii) In view of (8.2), $R > 0$ is preserved during the flow.

(iv) $\varepsilon R g_{ij} \le R_{ij} \le \beta R g_{ij}$ is preserved during the flow for $0 < \varepsilon \le \frac{1}{3} < \beta < 1$.

(v) Positive sectional curvatures are preserved.

(vi) The pinching estimate is a crucial estimate. It states that for some $\delta > 0$

$$(\lambda - \mu)^2 + (\lambda - \nu)^2 + (\mu - \nu)^2 \le cR^{2-\delta}.$$

It is the first estimate that is better than expected from scaling.

(vii) For any $\eta > 0$, we have the gradient estimate

$$|\nabla R|^2 \le \eta R^3 + c(\eta, g(0)).$$

(viii) As $t \nearrow T$, we have $\max\limits_M |\operatorname{Ric}| \to \infty$ and thus $\max\limits_M R \to \infty$.

(ix) From the gradient estimate, we obtain $\max R / \min R \to 1$ as $t \nearrow T$.

(x) Hence, after rescaling, $(M, g(t))$ converges to a manifold with $\lambda = \mu = \nu$ as $t \nearrow T$.

In contrast to mean curvature flow, as far as the results of this section are concerned, most of the a priori estimates rely on maximum principle.

Theorem 9.1 has been improved by C. Böhm and B. Wilking to any dimension and initial metrics of 2-positive curvature operator [3]. The main idea there is to carefully study the properties of the Riemannian curvature tensor in the ordinary differential equation related to Ricci flow and to introduce a continuous family of pinching cones. Ricci flow has to enter these cones transversally. Hence it has to converge to the intersection of all that cones, which consists of the metrics of constant (sectional) curvature.

This has been extended to manifolds M such that $M \times \mathbb{R}$ or $M \times \mathbb{R}^2$ has positive isotropic curvature (PIC) by S. Brendle and R. Schoen, see [4–6]. PIC is defined using orthonormal four-frames by the inequality

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0.$$

It implies in particular that a compact strictly 1/4-pinched Riemannian manifold is diffeomorphic to a space form, i. e. it implies the differentiable 1/4-pinching theorem. This result requires only pointwise strict 1/4-pinching for the sectional curvatures. Here, it is always understood that the sectional curvatures are positive.

The crucial step is to show that PIC is preserved under Ricci flow. It involves carefully studying the algebra in the corresponding ordinary differential equation.

10. Non-Compact Solutions

W.-X. Shi [42] has considered Ricci flow of non-compact complete manifolds of bounded curvature. He gives a short time existence proof in this situation. The proof relies on

(i) solving appropriate Dirichlet problems on a sequence of balls of increasing radius,

(ii) interior a priori estimates, some of them interior in space and time.

This is similar to what we have seen for graphical mean curvature flow. Note however, that for Ricci flow bounded curvature does not guarantee the existence of a solutions as neck-pinches may occur.

He has also proved non-compact maximum principles and has shown that manifolds that are initially Kähler stay Kähler. For those manifolds, the Kähler potential u fulfills (at least locally)

$$\dot{u} = \log\det u_{i\bar{j}}.$$

This flow is (currently) the main tool to attack the following conjecture of S.-T. Yau: A complete non-compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$.

The strategy is to let the metric evolve according to Kähler-Ricci flow. The biholomorphisms are then constructed with the help of the limit manifold (possibly after rescaling) as $t \to \infty$. This requires currently initially bounded curvature and at least one additional condition like average quadratic curvature decay

$$\frac{1}{|B_r(x)|}\int\limits_{B_r(x)} R\,d\mu \le \frac{c}{(1+r)^2} \quad \text{for all } x \in M,\ r > 0,$$

or maximal volume growth

$$\lim_{r\to\infty} \frac{|B_r(x)|}{r^n} > 0 \quad \text{for some } x \in M.$$

See the survey article of A. Chau and L.-F. Tam [8] for details.

Interior a priori estimates allow to reduce the smoothness that has to be assumed for the initial data. M. Simon [43] could start Ricci flow for C^0 initial data g_0, i. e. he constructs a smooth solution $g(t)$ that approximates g_0 in C^0 as $t \searrow 0$.

If g_0 is initially C^0-close to Euclidean space with the standard metric and the difference becomes small at infinity, the corresponding solutions were shown to converge to Euclidean space as $t \to \infty$ under Ricci harmonic map heat flow, see [38]. The proof is based on integral estimates involving

$$\sum_{i=1}^n \frac{1}{\lambda_i^m}(\lambda_i^m - 1)^2,$$

where $m \ge 2$ and λ_i denotes the eigenvalues of $g(t)$ with respect to the flat background metric.

11. Ricci Flow with Surgery

Ricci flow with surgery was introduced by R. Hamilton [25] for four-manifolds of positive isotropic curvature. Similar to mean curvature flow with surgery, it is important to understand precisely, how singularities look like in order to perform surgery. R. Hamilton made a lot of progress in this direction, but he could not exclude singularities that look locally like

$$\left(\mathbb{R}^2, \frac{1}{1+|x|^2}\delta\right) \times (\mathbb{R}, \delta),$$

a cigar soliton times $\mathbb{R}$, after blow-up. It is important to exclude such a soliton as it does not seem to allow for surgery.

G. Perelman could exclude such singularities. This is part of his recent work on Ricci flow of 3-manifolds [33–35]. Recent expositions of this material include [7, 10–12, 32, 46]. We will only sketch how to exclude such singularities.

Define for the metric g, a smooth function f and $\tau > 0$ the functional $\mathcal{W}$ by

$$\mathcal{W}(g, f, \tau) = \int \left[\tau \left(R + |\nabla f|^2\right) + f - n\right] u \, d\mu(g),$$

where $u = (4\pi\tau)^{-n/2} e^{-f}$ and n is the dimension of the manifold. We require that $\int u = 1$. This is called compatibility of u or f. Set

$$\mu(g, \tau) := \inf_f \mathcal{W}(g, f, \tau),$$

where the infimum is taken over all compatible functions f. This infimum is finite and attained for some smooth function f.

We have the following monotonicity: If M is closed and g, f, and τ evolve according to

$$\begin{cases} \frac{\partial g}{\partial t} = -2 \operatorname{Ric}, \\ \frac{d\tau}{dt} = -1, \\ \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \end{cases}$$

then

$$\frac{d}{dt} \mathcal{W}(g, f, \tau) = 2\tau \int \left| \operatorname{Ric} + \operatorname{Hess} f - \frac{g}{2\tau} \right|^2 u \, d\mu \geq 0.$$

This implies a lower bound on $\mu(g, \tau)$ during the flow. Such a lower bound implies (longer computations) a positive lower bound on the volume ratio for geodesic balls,

$$\frac{\mu(B_r(p))}{r^n} > \zeta > 0,$$

if $|\operatorname{Riem}| \leq r^{-2}$ on $B_r(p)$, where the lower bound depends on n, $g(0)$, and upper bounds for r and T.

Now we consider a special situation in dimension $n = 3$. If a solution $g(t)$ runs into a singularity as described above, we can choose small balls $B_r(p)$. If we fix the centers so that the balls lie on the region, where the cigar looks cylindrical, we obtain (after rescaling) the following situation. (Note that $|\operatorname{Riem}| \leq r^{-2}$ is preserved under scaling and so is the volume ratio.) The geodesic balls then roughly look like $\mathbb{S}^1 \times B_r \subset \mathbb{R}^2 \times \mathbb{R}^2$ and hence violate the lower volume bound for large values of r as the volume of these geodesic balls grows like r^2.

Appendix A. Parabolic Maximum Principles

The following maximum principle is fairly standard. For non-compact, strict or other maximum principles, we refer to [18] or [36], respectively.

We will use $C^{2,1}$ for the space of functions that have two spatial and one time derivative, if all these derivatives are continuous.

Theorem A.1 (Weak parabolic maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $T > 0$. Let a^{ij}, $b^i \in L^\infty(\Omega \times [0, T])$. Let a^{ij} be strictly elliptic, i. e. $a^{ij}(x, t) > 0$ in the sense of matrices. Let $u \in C^{2,1}(\Omega \times [0, T)) \times C^0\left(\overline{\Omega} \times [0, T]\right)$ fulfill*

$$\dot{u} \leq a^{ij} u_{ij} + b^i u_i \quad \text{in } \Omega \times (0, T).$$

Then we get for $(x,t)\in\Omega\times(0,T)$

$$u(x,t)\le\sup_{\mathcal{P}(\Omega\times(0,T))}u,$$

where $\mathcal{P}(\Omega\times(0,T)):=(\Omega\times\{0\})\cup(\partial\Omega\times(0,T))$.

Proof.

(i) Let us assume first that $\dot u<a^{ij}u_{ij}+b^iu_i$ in $\Omega\times(0,T)$. If there exists a point $(x_0,t_0)\in\Omega\times(0,T)$ such that $u(x_0,t_0)>\sup\limits_{\mathcal{P}(\Omega\times(0,T))}u$, we find $(x_1,t_1)\in\Omega\times(0,T)$ and t_1 minimal such that $u(x_1,t_1)=u(x_0,t_0)$. At (x_1,t_1), we have $\dot u\ge0$, $u_i=0$ for all $1\le i\le n$, and $u_{ij}\le0$ (in the sense of matrices). This, however, is impossible in view of the evolution equation.

(ii) Define for $0<\varepsilon$ the function $v:=u-\varepsilon t$. It fulfills the differential inequality

$$\dot v=\dot u-\varepsilon<\dot u\le a^{ij}u_{ij}+b^iu_i=a^{ij}v_{ij}+b^iv_i.$$

Hence, by the previous considerations,

$$u(x,t)-\varepsilon t=v(x,t)\le\sup_{\mathcal{P}(\Omega\times(0,T))}v=\sup_{\mathcal{P}(\Omega\times(0,T))}u-\varepsilon t$$

and the result follows as $\varepsilon\downarrow0$.

□

There is also a parabolic maximum principle for tensors, see [26, Theorem 9.1]. (See the AMS-Review for a small correction of the proof.)

A tensor N_{ij} depending smoothly on M_{ij} and g_{ij}, involving contractions with the metric, is said to fulfill the null-eigenvector condition, if $N_{ij}v^iv^j\ge0$ for all null-eigenvectors of M_{ij}.

Theorem A.2. *Let* M_{ij} *be a tensor, defined on a closed Riemannian manifold* $(M,g(t))$*, fulfilling*

$$\frac{\partial}{\partial t}M_{ij}=\Delta M_{ij}+b^k\nabla_kM_{ij}+N_{ij}$$

on a time interval $[0,T)$*, where* b *is a smooth vector field and* N_{ij} *fulfills the null-eigenvector condition. If* $M_{ij}\ge0$ *at* $t=0$*, then* $M_{ij}\ge0$ *for* $0\le t<T$*.*

Appendix B. Some linear algebra

Lemma B.1. *We have*

$$\frac{\partial}{\partial a_{ij}}\det(a_{kl})=\det(a_{kl})a^{ji},$$

if a_{ij} *is invertible with inverse* a^{ij}*, i. e. if* $a^{ij}a_{jk}=\delta^i_k$.

Proof. It suffices to prove that the claimed inequality holds when we multiply it with a_{ik} and sum over i. Hence, we have to show that

$$\frac{\partial}{\partial a_{ij}}\det(a_{kl})a_{ik}=\det(a_{kl})\delta^j_k.$$

We get

$$\frac{\partial}{\partial a_{ij}} \det(a_{kl}) = \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & 0 & a_{1\,j+1} & \dots & a_{1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1\,1} & \dots & a_{i-1\,j-1} & 0 & a_{i-1\,j+1} & \dots & a_{i-1\,n} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ a_{i+1\,1} & \dots & a_{i+1\,j-1} & 0 & a_{i+1\,j+1} & \dots & a_{i+1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix}.$$

and thus

$$\begin{aligned} \frac{\partial}{\partial a_{ij}} \det(a_{kl}) \cdot a_{ik} =& \det \begin{pmatrix} 0 & \dots & 0 & a_{1\,k} & 0 & \dots & 0 \\ a_{2\,1} & \dots & a_{2\,j-1} & 0 & a_{2\,j+1} & \dots & a_{2\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & 0 & a_{1\,j+1} & \dots & a_{1\,n} \\ 0 & \dots & 0 & a_{2\,k} & 0 & \dots & 0 \\ a_{3\,1} & \dots & a_{3\,j-1} & 0 & a_{3\,j+1} & \dots & a_{3\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \dots \\ =& \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & a_{1\,k} & a_{1\,j+1} & \dots & a_{1\,n} \\ a_{2\,1} & \dots & a_{2\,j-1} & 0 & a_{2\,j+1} & \dots & a_{2\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & 0 & a_{1\,j+1} & \dots & a_{1\,n} \\ a_{2\,1} & \dots & a_{2\,j-1} & a_{1\,k} & a_{2\,j+1} & \dots & a_{2\,n} \\ a_{3\,1} & \dots & a_{3\,j-1} & 0 & a_{3\,j+1} & \dots & a_{3\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \dots \\ =& \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & a_{1\,k} & a_{1\,j+1} & \dots & a_{1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & a_{n\,k} & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ =& \delta^j_k \det(a_{rs}). \end{aligned}$$

□

Lemma B.2. *Let $a_{ij}(t)$ be differentiable in t with inverse $a^{ij}(t)$. Then*

$$\frac{d}{dt} a^{ij} = -a^{ik} a^{lj} \frac{d}{dt} a_{kl}.$$

Proof. We have

$$a^{ik} a_{kj} = \delta^i_j.$$

Assume that there exists $\tilde{a}^{ij}$ such that

$$a_{ik}\tilde{a}^{kj} = \delta_i^j.$$

Then $a^{ij} = \tilde{a}^{ij}$, as

$$a^{ij} = a^{ik}\delta_k^j = a^{ik}\left(a_{kl}\tilde{a}^{lj}\right) = \left(a^{ik}a_{kl}\right)\tilde{a}^{lj} = \tilde{a}^{ij}.$$

We differentiate and obtain

$$0 = \frac{d}{dt}\delta_j^i = \frac{d}{dt}\left(a^{ik}a_{kj}\right) = \frac{d}{dt}a^{ik}a_{kj} + a^{ik}\frac{d}{dt}a_{kj}.$$

Hence

$$\frac{d}{dt}a^{il} = \frac{d}{dt}a^{ik}\delta_k^l = \frac{d}{dt}a^{ik}a_{kj}a^{jl} = -a^{ik}\frac{d}{dt}a_{kj}a^{jl}.$$

□

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