Tropical Geometry

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There are many examples in algebraic geometry in which complicated geometric or algebraic problems can be transformed into purely combinatorial ones. The most prominent example is probably given by toric varieties — a certain class of varieties that can be described purely by combinatorial data, e.g. by giving a convex polytope in an integral lattice. As a consequence, most questions about these varieties can be transformed into combinatorial questions on the defining polytope that are then hopefully easier to solve.

Tropical algebraic geometry is a recent development in the field of algebraic geometry that tries to generalize this idea substantially. Ideally, every construction in algebraic geometry should have a combinatorial counterpart in tropical geometry. One may thus hope to obtain results in algebraic geometry by looking at the tropical (i.e. combinatorial) picture first and then trying to transfer the results back to the original algebro-geometric setting.

The origins of tropical geometry date back about twenty years. One of the pioneers of the theory was Imre Simon, a mathematician and computer scientist from Brazil — which is by the way the only reason for the peculiar name “tropical geometry”. Originally, the theory was developed in an applied context of discrete mathematics and optimization, but it has not been part of the mainstream in either of mathematics, computer science, or engineering. Only in the last few years have people realized its power for applications in fields such as combinatorics, computational algebra, and algebraic geometry. For example, Mikhalkin has proven recently that tropical geometry can be used to compute the numbers of plane curves of given genus g and degree d through $3d+g-1$ general points — a deep result that had been obtained first by Caporaso and Harris about ten years ago by a complicated study of moduli spaces of plane curves.

In my series of lectures in Alpbach I will give an introduction to the theory of tropical geometry, emphasizing the connections with and applications to algebraic geometry. Our main focus will be on enumerative, i.e. “curve counting” problems like the one mentioned above, since this is one of the areas of mathematics on which tropical geometry had the most impact recently.