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HEIGHT BOUNDS IN DIOPHANTINE GEOMETRY

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Heights of algebraic points on varieties play an important role in diophantine geometry. Height upper bounds were used for example in the proof of the Mordell-Lang Conjecture by Faltings, McQuillan, and Vojta.

In 1999, Bombieri, Masser, and Zannier studied the intersection of a curve C in the algebraic torus $\mathbf{G}_m^n$ with algebraic subgroups. If C is defined over the field of algebraic numbers and if it is not contained in the translate of a proper algebraic subgroup, then they proved that a point on C contained in a proper algebraic subgroup has height bounded independently of the subgroup. They used this height bound, together with Lehmer-type height lower bounds, to prove that there are only finitely many points on C contained in an algebraic subgroup of codimension at least 2.

Bombieri, Masser, and Zannier stated several conjectures on the intersection of subvariety of the algebraic torus with the union of all algebraic subgroups of fixed dimension. Let A be a semi-abelian variety and let X an irreducible subvariety of A. After defining an appropriate height on A the Bounded Height Conjecture expects that a point on $X\backslash Z$ contained in an algebraic subgroup of A of codimension at least $\dim X$ has height bounded independently of the subgroup. Here $Z \subset X$ is the union of "anomalous" subvarieties of X. Motivated by the original result on curves, one conjectures that there are only finitely points on $X\backslash Z$ contained in an algebraic subgroup of codimension at least $1 + \dim X$. In fact one expects finiteness with Z replaced by the possibly smaller union of all "torsion anomalous" subvarieties Z'. At least conjecturally, Z and Z' are Zariski closed. A finiteness conjecture like the one above was stated independently and in somewhat different terms by Pink and Zilber. In its strongest form, the finiteness conjecture implies the Mordell-Lang Conjecture which is known to be true.

The lectures begin with a survey of results related to the Bounded Height Conjecture due to Bombieri, Masser, and Zannier, Maurin, Rémond, Viada, and the lecturer. Next we introduce relevant concepts likes heights. We then outline a proof of the Bounded Height Conjecture when A is a power of an elliptic curve. The proof uses intersection theory, in particular Siu's Theorem, and a Theorem of Ax on analytic subgroups of commutative algebraic groups. Finally, we will see how to deduce finiteness from boundedness of height if a suitable Lehmer-type height lower bound for A is known. By a recent result of Ratazzi this is the case if A is a power of an elliptic curve with complex multiplication.